Control and Electric Circuits

Prepared by

Prof. Dr. Mohamed EL-Shimy

Professor of electric power systems - Department of Electrical Power and Machines - Faulty of Engineering – Ain Shams University

Dr. Mahmoud Abdallah Attia

Department of Electrical Power and Machines - Faulty of Engineering – Ain Shams University

Second Year

2019

Acknowledgments

This two-year curriculum was developed through a participatory and collaborative approach between the Academic faculty staff affiliated to Egyptian Universities as Alexandria University, Ain Shams University, Cairo University , Mansoura University, Al-Azhar University, Tanta University, Beni Souef University , Port Said University, Suez Canal University and MTI University and the Ministry of Health and Population(General Directorate of Technical Health Education (THE). The design of this course draws on rich discussions through workshops. The outcome of the workshop was course specification with Indented learning outcomes and the course contents, which served as a guide to the initial design.

We would like to thank **Prof.Sabah Al- Sharkawi** the General Coordinator of General Directorate of Technical Health Education, **Dr. Azza Dosoky** the Head of Central Administration of HR Development, **Dr. Seada Farghly** the General Director of THE and all share persons working at General Administration of the THE for their time and critical feedback during the development of this course.

Special thanks to the **Minister of Health and Population Dr. Hala Zayed and Former Minister of Health Dr. Ahmed Emad Edin Rady** for their decision to recognize and professionalize health education by issuing a decree to develop and strengthen the technical health education curriculum for pre-service training within the technical health institutes.

Contents

توصيف مقرر دراسى

1- بيانات المقرر

الرمز الكودى :	اسم المقرر :التحكم والدوائر الكهربائية	الفرقة /المستوى : السنة الثانية
التخصص :	عدد الوحدات الدراسية : 2 نظرى 2 عملى	

2- هدف المقرر:

To Describe the fundamentals of electric control circuits, their performances, and their designs. In addition, to be able to select a suitable control system for a specific application.

3- المستهدف من تدريس المقرر :

Students of Technical Health Institutes.

ا. المعلومات والمفاهيم :

By the end of this course, students should be able to:

- Describe various electronic switches used in industrial applications.
- Recognize the differences between open and closed loop control systems.
- Identify the characteristics of various electronic conversion circuits for power supply control applications.
- Describe the characteristics and applications of various motor types.
- Elaborate on the control circuit requirements, performances, and designs.

ب- المهارات الذهنية :

By the end of this course, students should be able to:

- Analyze the performance of each AC/DC rectification circuit.
- Compare between the performances of various AC/DC rectification circuits.
- Analyze the performance of each AC wave controlled circuit.
- Compare between the performances of various AC wave controlled circuits.
- Analyze the performance of each DC/AC inversion circuit.
- Compare between the performances of various DC/AC inversion circuits.

• Evaluate of performances of various motor types.	
By the end of this course, students should be able to: • Measure the performances of various AC and DC switching circuits. • Select appropriate motor for a specific application based on the desired performance. • Inspect drawings of electric and electronic control circuits and schematic diagrams. • Test and set PID control parameters.	ج- المهارات المهنية الخاصة بالمقرر:
By the end of this course, students should be able to: • Work in a team to perform specific targets such as preparing a report or a project. • Write technical report. • Use communication skills through various media types.	د- المهارات العامة :
1. Introduction to electricity. (4 hours lecture) 2. Basics of power electronics: (12 hours lecture + 12 hours lab) a. Basic electronic switches. b. Uncontrolled rectification circuits. c. Controlled rectification circuits. d. AC control circuits. e. DC/AC inversion circuits. 3. Electric motors for industrial applications: (8 hours lecture + 8 hours lab) a. DC motors and their types. b. AC induction motors. c. AC synchronous motors. d. Stepper motors. 4. Basic control circuits: (4 hours lecture + 4 hours lab) a. Op b. en loop control. c. Closed loop control. d. PID controllers. 5. Review. (2 hours)	4- محتوى المقرر:

Lectures **Site visits.** **Laboratory experiments.** **Interactive learning.**	5- أساايب التعليم والتعلم
Individual guidance. Individual feedback.	6- أساليب التعليم والتعلم للطلاب ذوى القدرات المحدودة
	7- تقويم الطلاب :
• Assignments • Periodic quizzes • Midterm exam • Final exam	أ- الأساليب المستخدمة
• Occasional assignments and oral discussions. • Periodic quizzes: weeks 3, 6, 10, 12 • Midterm exam: week 7 • Final exam: end of term	ب- التوقيت
• Semester work 10% • Practical work 10% • Final written exam 80%	ج- توزيع الدرجات

8- قائمة الكتب الدراسية والمراجع :

K. Ogata, "Modern control engineering",
M.H. Rashid, "Power electronics: circuits, devices, and applications", 2009.

Special notes and presentations prepared for this course.	أ- مذكرات

Course handbook	ب- كتب ملزمة
K. Ogata, "Modern control engineering", M.H. Rashid, "Power electronics: circuits, devices, and applications", 2009.	ج- كتب مقترحة
IEEE periodicals http://shimymb.tripod.com	د- دوريات علمية أو نشرات ...... الخ

Course Description

This book presents an integrated lumped course targeting the electric circuits, and control systems. The main objective is to provide the students the capability of dealing with the general control problem with focus on practical electrical systems. Consequently, the modeling and analysis of basic DC and AC electric circuits are presented in this book. In addition, the basic electronic switching circuits used in industrial applications are also presented. The control of electric systems which is the main core of this book is given a significant focus.

The book consists of six chapters. The first chapter presents an introduction to control systems. The objectives of this chapter include the identification of the requirements of control systems. In addition, the chapter presents an overview of the fundamentals of control systems used in industrial applications. The second chapter presents the fundamental of DC electric circuits, and the basic theories for their modeling, and analysis. The third chapter handles the single-phase AC electric circuits, while the fourth chapter presents the electronic switches, and the single-phase switching circuits used in industrial and power applications. Chapter 5 presents the modeling, and representation of general control systems with focus on practical electric circuits, and systems. The last chapter presents block diagrams representation of control systems, and examples of the analysis of responses of control systems.

Core Knowledge

By the end of this course, students should be able to:

- Define the functions of various elements of electric circuits and control systems.
- List various types of control systems.
- List various types of power electronic switches used in industrial applications and control systems.

Core Skills

By the end of this course, students should be able to:

- Describe the basic control system requirements.
- Give practical examples of the applications of automatic control systems.
- Model and solve DC circuits.

- Model and solve single-phase AC circuits.
- Model and solve basic single-phase power switching circuits,.
- Devise the structure of control systems for specific control targets.
- Solve Laplace transform, and inverse Laplace transform problems.
- Determine the transfer functions of electric circuits, and some industrial systems.
- Represent physical systems by mathematical models for use in control system representation.
- Represent control systems using block diagrams.
- Analyze the dynamic response of first order systems.

Course Contents

		Methods of Teaching/Training with Number of Total Hours per Topic				
ID	**Topics**	**Interactive Lecture**	**Field Work**	**Class Assignments**	**Research**	**Lab**
1	Introduction to control systems	4		2		
2	DC Electric Circuits	2		2		2
3	Single-phase AC Electric Circuits	2		2		2
4	Power Electronic Switching Circuits	2		4		2
5	Control System Modeling and Representation	6		4		2
6	Block Diagrams and Analysis of the Responses	4	6	4		
	TOTAL HOURS (52)	**20**	**6**	**18**		**8**

Chapter 1

Introduction to Control Systems

Objectives

The objectives of this chapter include the identification of the requirements of control systems. In addition, the chapter presents an overview of the fundamentals of control systems used in industrial applications.

Introduction

The *control systems* are formed by integrating elements. The function of these elements is to maintain a process variable at a desired value or within a desired range of values. Requiring the human operator to take all of the required corrective action manually is impractical, or sometimes impossible, especially if a large number of indications must be monitored simultaneously, and fast actions should be taken at appropriate times. For these reasons, most systems are controlled automatically once they are operating under normal conditions. Automatic controls greatly reduce the burden on the operator and make his or her job manageable. Process variables requiring control in a system include, but are not limited to, flow, level, temperature, pressure, voltage magnitude, and power flow. Some systems do not require all of their process variables to be fully controlled.

Automatic control systems, neither replace nor relieve the operator of the responsibility for maintaining the facility by performing fully automatic actions. The operation of the control systems is periodically checked by the operators to verify proper operation. If a control system fails, the operator must be able to take over and control the process manually. In most cases, understanding how the control system works aids the operator in determining the system is operating properly and which actions are required to maintain the system in a safe condition. Automatic control systems as essential parts in many applications such as various industries, transport, and household equipment.

Historical Timeline of Control Systems

- James Watt's centrifugal governor for the speed control of a steam engine, in the eighteenth century.
- In 1922, Minorsky worked on automatic controllers for steering ships, and showed how stability could be determined from the differential equations describing the system.
- In 1932 Nyquist developed a relatively simple procedure for determining the stability of a closed-loop system.
- In 1934 Hasen introduced the term "servomechanisms" for position control systems, and designed the relay servomechanisms, capable of closely following a changing input.

- During the 1940's frequency-response methods were used to design linear feedback control systems.
- From the end of the 1940's to early 1950's, the root-locus method in control system design was fully developed.
- Since the late 1950's, the emphasis in control system design concentrated on the design of optimal control systems.
- Since about 1960, modern control systems with multi-input-multi-output variables were developed.
- Digital control systems emerged in the late 1970's.
- Highly developed computer control systems started since the early 1980's.

Definitions and Terminology

- *Plants*. A plant is a piece of equipment, or set of machine parts, or a set of machines which performs a particular operation(s). It is the object to be controlled.
- *Processes*. A process may be defined as a natural, progressive continuing operation that leads toward a particular result or end.
- *Systems*. A system is a combination of components that act together and perform a certain objective.
- *Disturbances*. A disturbance is a signal which tends to adversely affect the value of the output of the system. There are *internal* and *external* disturbances.
- *Control system*. A control system is a system of integrated elements whose function is to maintain a process variable at a desired value or within a desired range of values. The control system monitors a process variable or variables, then causes some action to occur to maintain the desired system parameter. In the example of the central heating unit, the system monitors the temperature of the house using a thermostat. When the temperature of the house drops to a preset value, the furnace turns on, providing a heat source. The temperature of the house increases until a switch in the thermostat causes the furnace to turn off.
- *Control system input*. The control system input is the stimulus applied to a control system from an external source to produce a specified response from the control system. In the case of the central heating unit, the control system input is the temperature of the house as monitored by the thermostat.
- *Control system output*. Control system output is the actual response obtained from a control system. In the example above, the temperature dropping to a preset value on the thermostat causes the furnace to turn on, providing heat to raise the temperature of the house.
- *Feedback control*. Feedback control is an operation which, in the presence of disturbances, tends to reduce the difference between the output of a system and the reference input, and which does so, on the basis of this difference.
- *Feedback control systems*. A feedback control system is one which tends to maintain a prescribed relationship between the output and the reference input by comparing these and using the difference as a means of control.
- *Servomechanisms*. A servomechanism is a feedback control system in which the output is some mechanical position, velocity, or acceleration.

- *Automatic regulating systems*. An automatic regulating system is a feedback control system in which the reference input or the desired output is either constant or slowly varying with time, and in which the primary task is to maintain the actual output at the desired value despite the presence of disturbances.
- *Process control systems*. An automatic regulating system in which the output is a variable such as temperature, pressure, flow, or liquid level is called by a *process control system*.
- *Controlled variable*. A controlled variable is the process variable that is maintained at a specified value or within a specified range. In the previous example, the storage tank level is the controlled variable.
- *Manipulated variable*. A manipulated variable is the process variable that is acted on by the control system to maintain the controlled variable at the specified value or within the specified range. In the previous example, the flow rate of the water supplied to the tank is the manipulated variable.

Closed-Loop Control Systems and Open-Loop Control

Control systems are classified by the controller action, which is the quantity responsible for activating the control system to produce the output. The two general classifications are open-loop and closed-loop control systems.

An *open-loop control system* is one in which the control action is independent of the output (see Fig. 1). Any control system which operates on a time basis is open-loop. Examples: washing machine, traffic lights… etc. A *closed-loop control system* is one in which control action is dependent on the output as shown in Fig. 2.

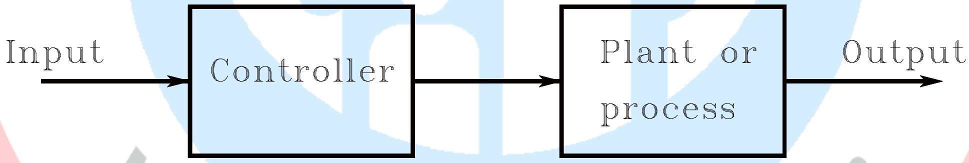


Fig. 1: Open-loop control systems

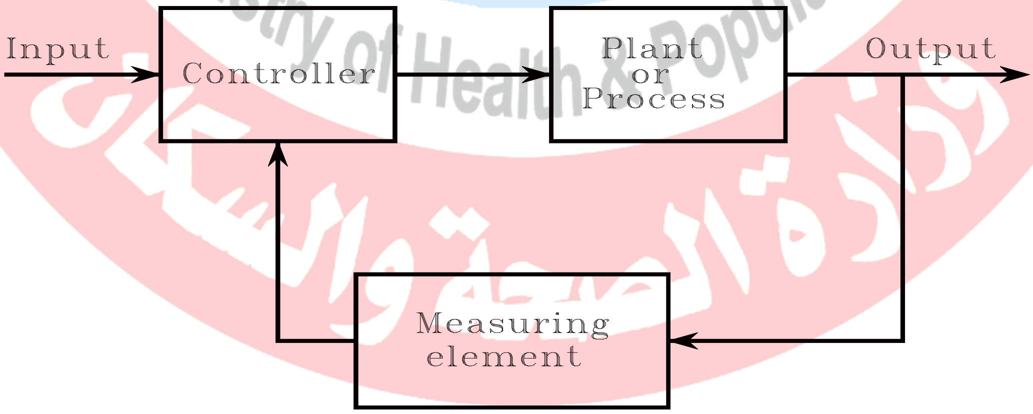


Fig. 2: Closed-loop control system.

An advantage of the closed-loop control system is that the use of feedback makes the system response relatively insensitive to external disturbances or internal variations in system parameters. From the point of view of stability, the open-loop control system is easier to build since stability is not a major problem. On the other hand, stability is always a major problem in the closed-loop control system since it may tend to overcorrect error which may cause oscillations of constant or changing amplitude. A proper combination of open-loop and closed-loop controls will usually less expensive and give satisfactory overall system performances. Figs. 3, and 4 illustrate the open-loop, and closed-loop control of a thermal system. The reader is required to explain the functions of these control options.

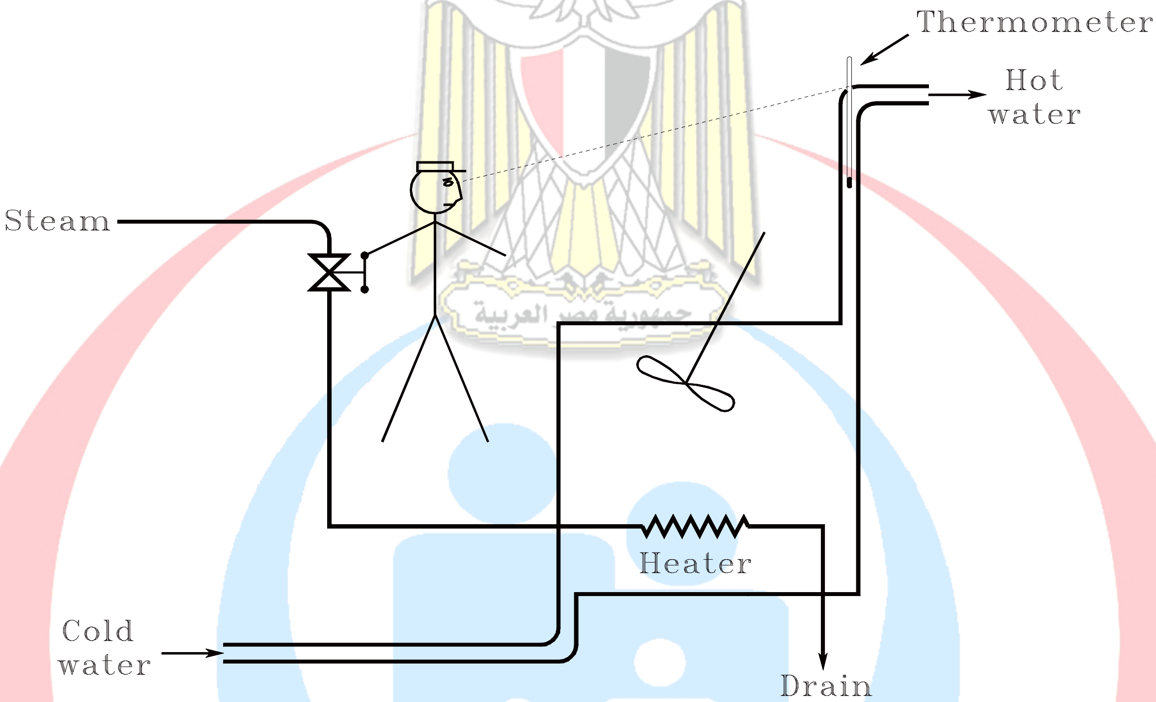


Fig. 3. Manual feedback control of a thermal system.

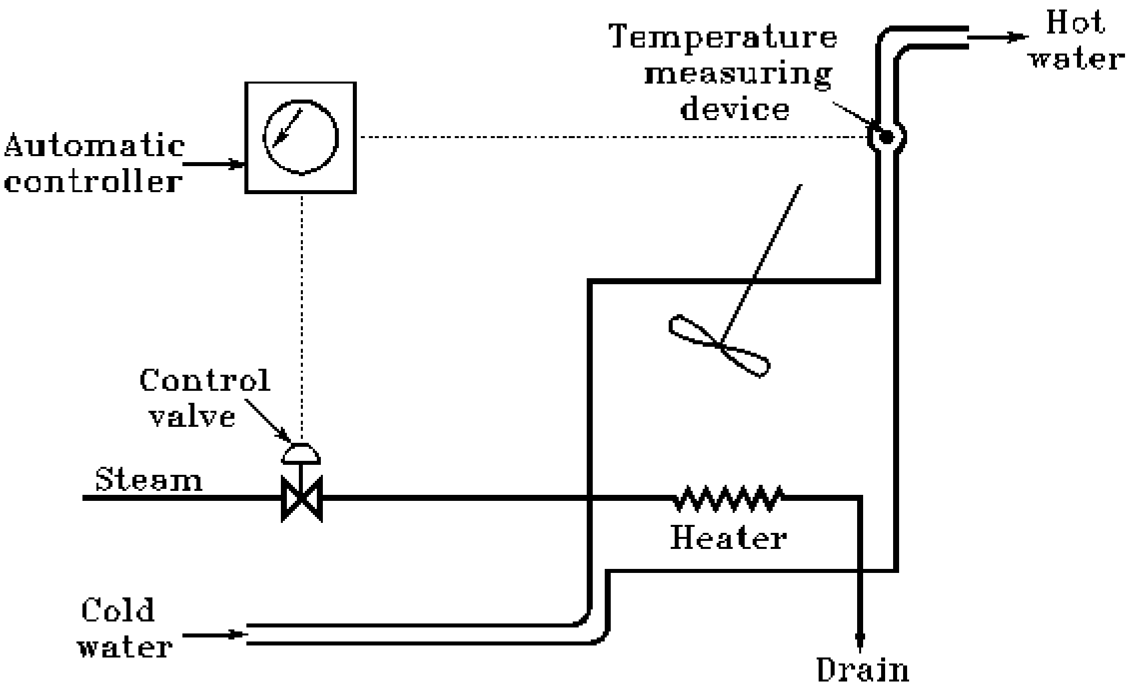


Fig. 4: Automatic feedback control of a thermal system.

Functions of Automatic Control

In any automatic control system, the four basic functions that occur are:

1. Measurement
2. Comparison
3. Computation
4. Correction

In the water tank level control system in the example above, the level transmitter measures the level within the tank. The level transmitter sends a signal representing the tank level to the level control device, where it is compared to a desired tank level. The level control device, then computes how far to open the supply valve to correct any difference between actual and desired tank levels.

Elements of Automatic Control

The three functional elements needed to perform the functions of an automatic control system are:

1. A measurement element
2. An error detection element
3. A final control element

Relationships between these elements and the functions they perform in an automatic control system are shown in Fig. 5. The measuring element performs the measuring function by sensing and evaluating the controlled variable. The error detection element first compares the value of the controlled variable to the desired value, and then signals an error if a deviation exists between the actual and desired values. The final control element responds to the error signal by correcting the manipulated variable of the process.

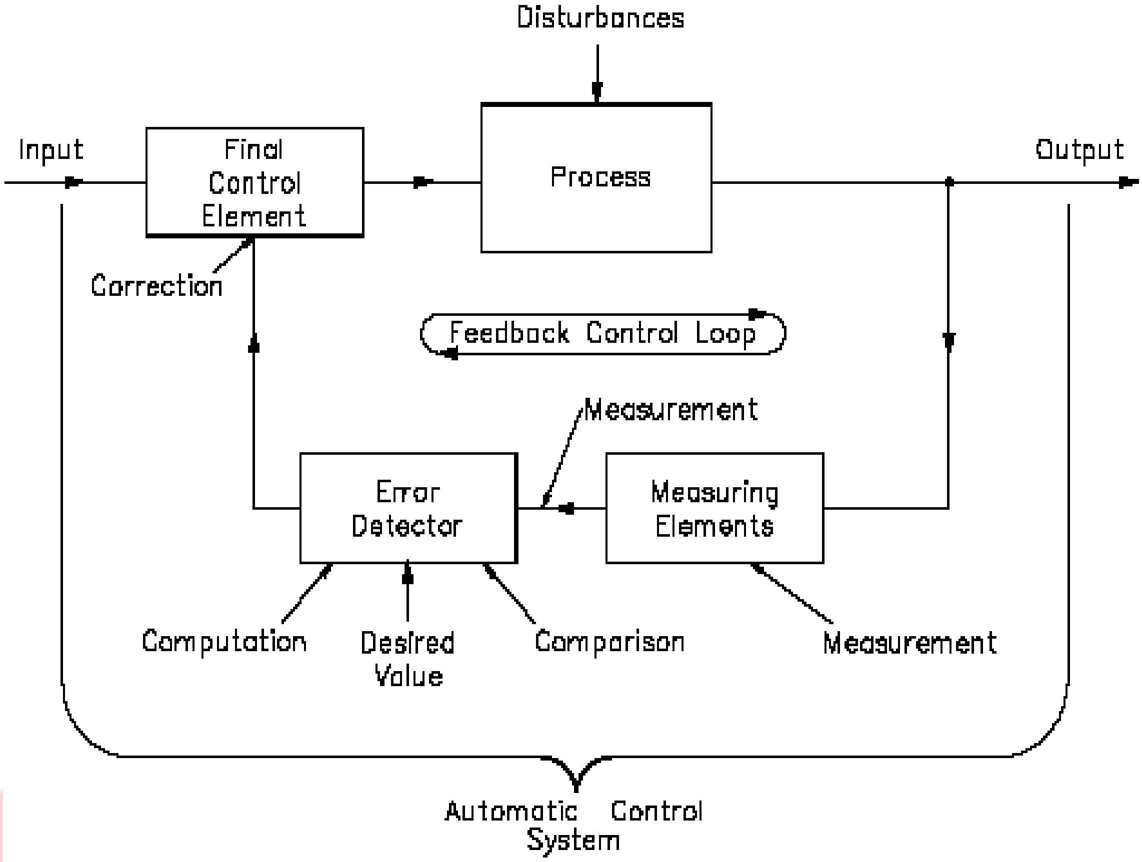


Fig. 5: Relationships between functions, and elements in an automatic control system

An automatic controller is an error-sensitive, self-correcting device. It takes a signal from the process and feeds it back into the process. Therefore, closed-loop control is referred to as feedback control.

Direct versus indirect controls

It is desirable to measure and control directly the variable which indicates the state of the system or the quality of the product. This may present a difficult problem, since this quality may be difficult to measure. If this is the case, it becomes necessary to control a secondary variable (such as temperature and pressure). Although it may be difficult, we should always try to control the primary variable as directly as possible.

Adaptive control systems

The dynamic characteristics of most control systems are not constant because of possible variations in their parameters or environment. (e.g. variation in mass, or temperature … etc.). In such cases a satisfactory system must have the ability of adaptation. Adaptation implies the ability to self-modify or self-adjust in accordance with unpredictable changes in conditions of environment or structure. These are called “adaptive control systems”.

Learning and intelligent control systems

Many apparently open-loop control systems can be converted into closed-loop control system if a human operator is considered as a controller. (e.g. train driver). As the operator gains more experience, he or she will become a better controller. Recently learning and intelligent systems are being developed.

Illustrative Examples of Control Systems

- **Pressure control systems; Fig. 6**

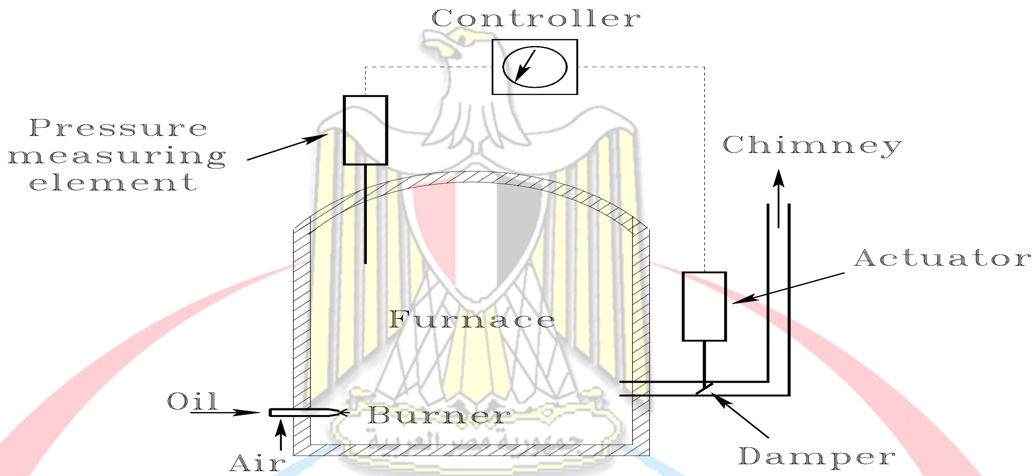


Fig. 6: Pressure control system.

- **Speed control systems; Fig. 7**

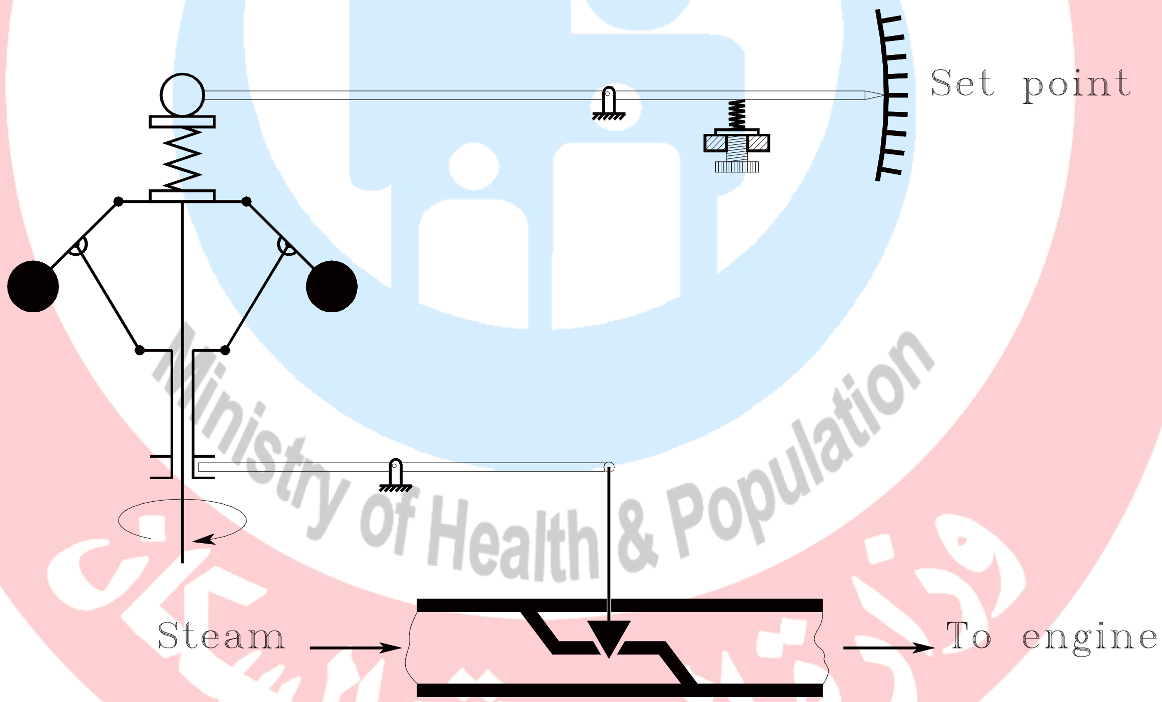


Fig. 7: Speed control system.

- **Computer control systems; Fig. 8**

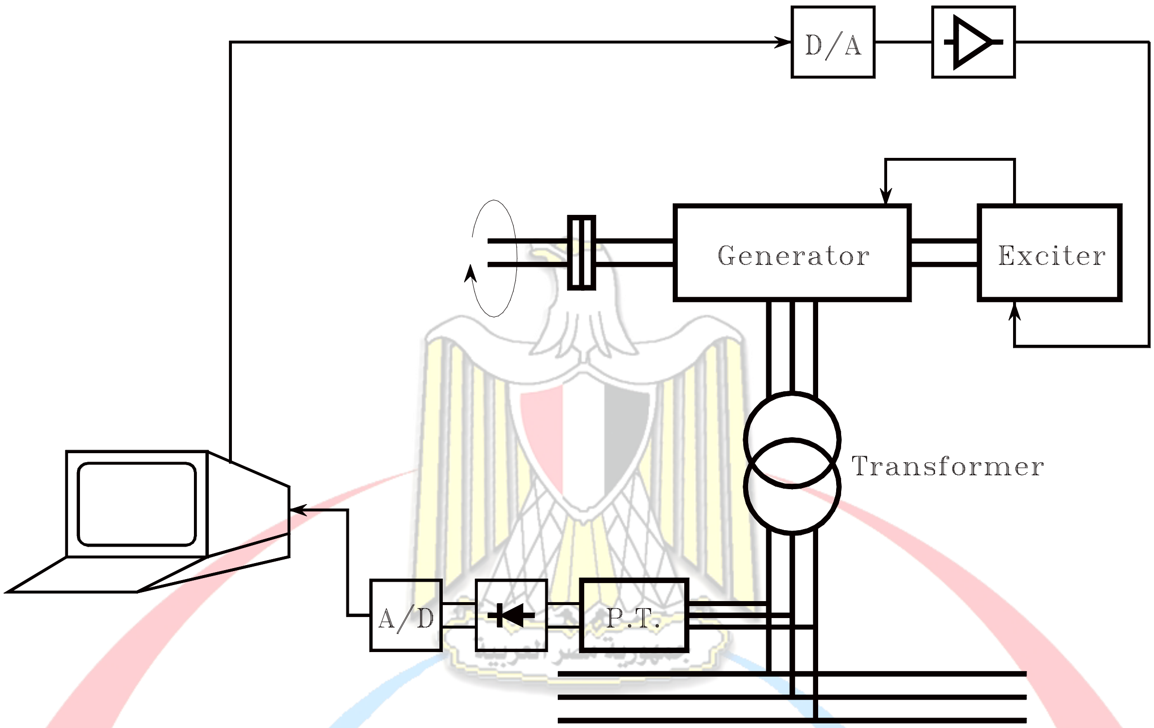


Fig. 8: Computer control system.

Control Loop Diagrams

A ***loop diagram*** is a "roadmap" that traces the process flow through the system and designates variables that can disrupt the balance of the system. A ***block diagram*** is a pictorial representation of the cause and effect relationship between the input and output of a physical system. A block diagram provides a mean to easily identify the functional relationships among the various components of a control system.

The simplest form of a block diagram is the *block and arrows diagram*. It consists of a single block with one input and one output (Figure 9 (A)). The block normally contains the name of the element (Figure 9(B)) or the symbol of a mathematical operation (Figure 9(C)) to be performed on the input to obtain the desired output. Arrows identify the direction of information or signal flow.

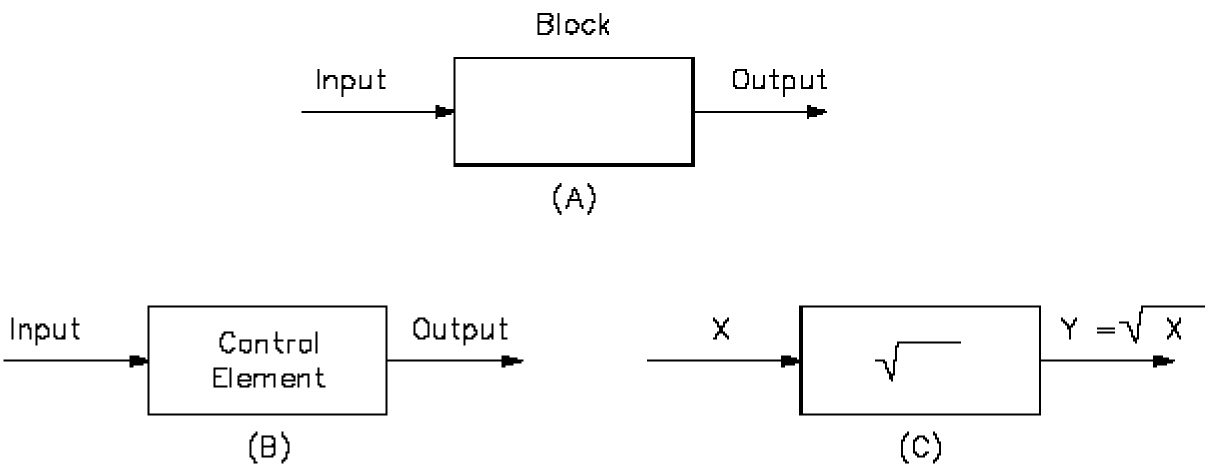


Fig. 9: Blocks, and arrows

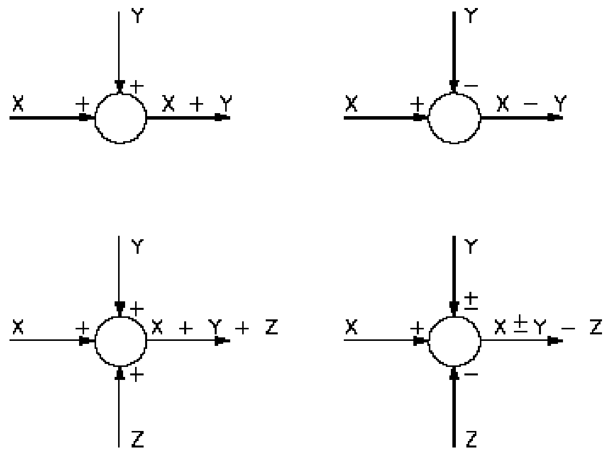


Fig. 10: Summation points

Although blocks are used to identify many types of mathematical operations, operations of addition and subtraction are represented by a circle, called a *summing point*. As shown in Figure 10, a summing point may have one or several inputs. Each input has its own appropriate plus or minus sign. A summing point has only one output and is equal to the algebraic sum of the inputs.

A *takeoff point* is used to allow a signal to be used by more than one block or summing point (Figure 11).

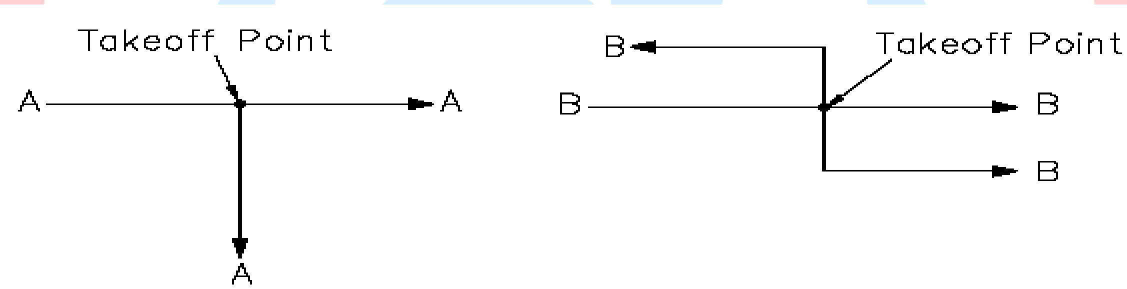


Fig. 11: Takeoff points

Feedback Control system Block Diagram

Figure 12 shows the basic elements of a feedback control system as represented by a block diagram. The functional relationships between these elements are easily seen. An important factor to remember is that the block diagram represents flow paths of control signals, but does not represent the flow of energy through the system or process.

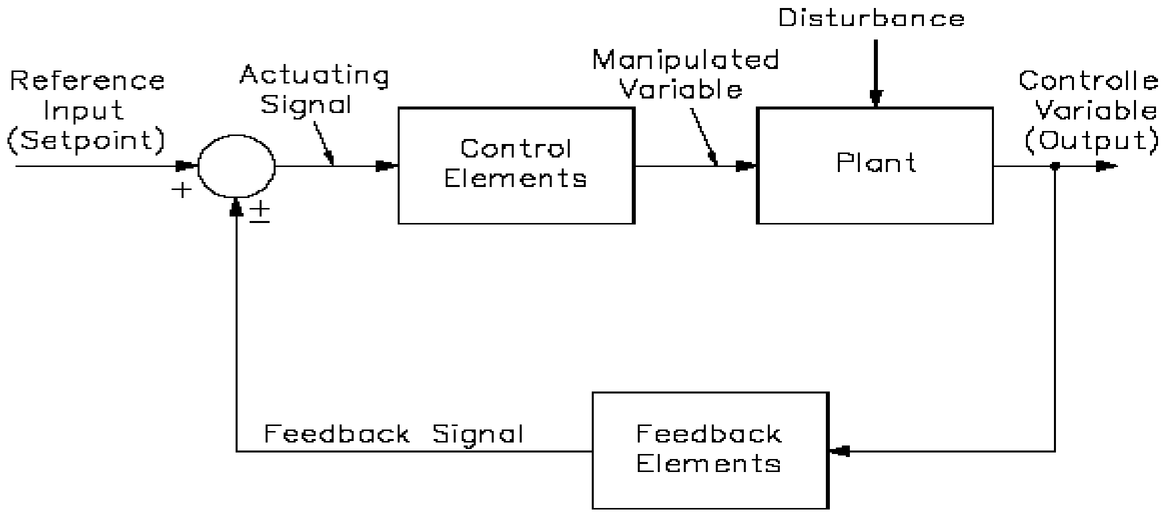


Fig. 12: Feedback control system block diagram

Figure 13 shows a typical application of a block diagram to identify the operation of a temperature control system for lubricating oil.

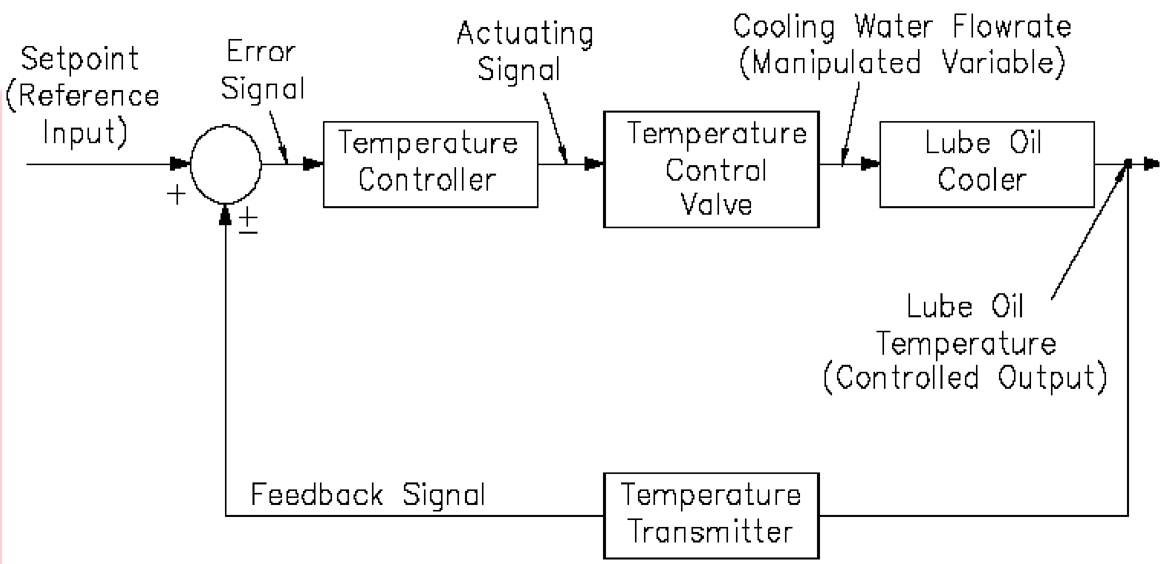


Fig. 13: Lubrication oil cooler – the block diagram of the temperature control system

Process Time Lag

In the last example, the control of the lubrication oil temperature may initially seem easy. Apparently, the operator need only measure the lube oil temperature, compare the actual temperature to the desired (set point), compute the amount of error (if any), and adjust the temperature control valve to correct the error accordingly. However, processes have the characteristic of delaying and retarding changes in the values of the process variables. This characteristic greatly increases the difficulty of control.

Process time lags are the general term that describes these process delays and retardations. Process time lags are caused by three properties of the process. They are: *capacitance*, *resistance*, and *transportation time*.

Capacitance is the ability of a process to store energy. In Figure 13, for example, the walls of the tubes in the lube oil cooler, the cooling water, and the lube oil can store heat energy. This energy-storing property gives the ability to retard change. If the cooling water flow rate is increased, it will take a period of time for more energy to be removed from the lube oil to reduce its temperature.

The resistance is part of the process that opposes the transfer of energy between capacities. In Figure 13, the walls of the lube oil cooler oppose the transfer of heat from the lube oil inside the tubes to the cooling water outside the tubes.

Transportation time is the time required to carry a change in a process variable from one point to another in the process. If the temperature of the lube oil (Figure 13) is lowered by increasing the cooling water flow rate, some time will elapse before the lube oil travels from the lube oil cooler to the temperature transmitter. If the transmitter is moved farther from the lube oil cooler, the transportation time will increase. This time lag is not just a slowing down or retardation of a change; it is an actual time delay during which no change occurs.

Stability of Automatic Control System

All control modes previously described can return a process variable to a steady value following a disturbance. This characteristic is called stability. *Stability* is the ability of a control loop to return a controlled variable to a steady, non-cyclic value, following a disturbance. Control loops can be either stable or unstable. Instability is caused by a combination of process time lags discussed earlier (i.e., capacitance, resistance, and transport time) and inherent time lag within a control system. This results in slow response to changes in the controlled variable. Consequently, the controlled variable will continuously cycle around the setpoint value.

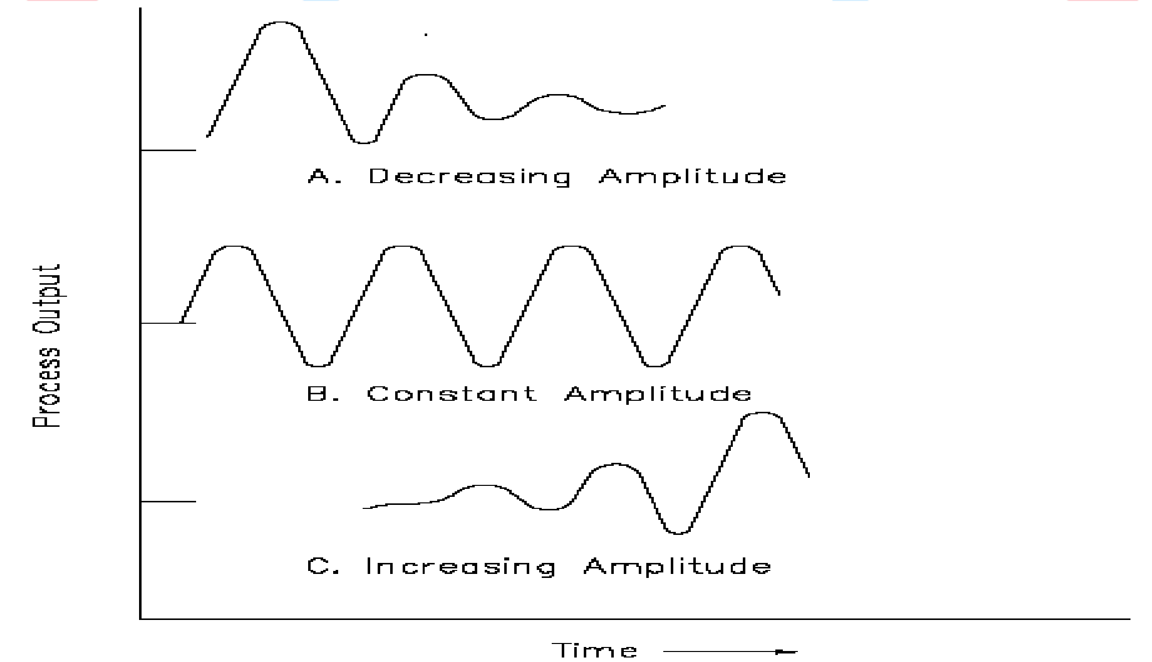


Fig. 14: Types of oscillations

Oscillations describe this cyclic characteristic. There are three types of oscillations that can occur in a control loop. They are *decreasing amplitude, constant amplitude*, and *increasing amplitude*. Each is shown in Figure 14.

Decreasing amplitude (Figure 14(A)). These oscillations decrease in amplitude and eventually stop with a control system that opposes the change in the controlled variable. This is the condition desired in an automatic control system.

Constant amplitude (Figure 14(B)). The action of the controller sustains oscillations of the controlled variable. The controlled variable will never reach a stable condition; therefore, this condition is not desired.

Increasing the amplitude (Figure 14(C)). The control system not only sustains oscillations, but also increases them. The control element has reached its full travel limits and causes the process to go out of control.

Two position Control Systems

A two position controller is the simplest type of controller. A controller is a device that generates an output signal based on the input signal it receives. The input signal is actually an error signal, which is the difference between the measured variable and the desired value, or setpoint; Figure 15.

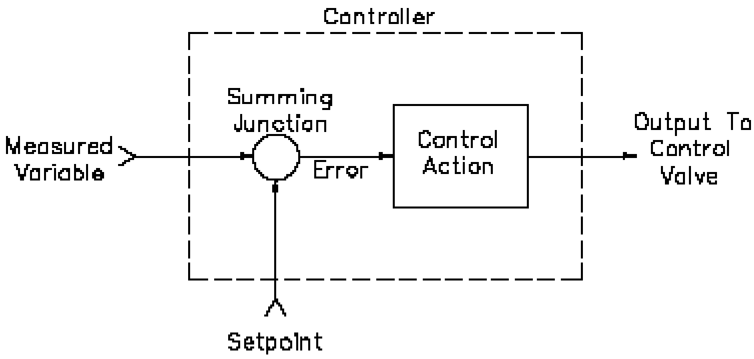


Fig. 15: Process control system operation

This input error signal represents the amount of deviation between where the process system is actually operating and where the process system is desired to be operating. The controller provides an output signal to the final control element, which adjusts the process system to reduce this deviation. The characteristic of this output signal is dependent on the type, or mode, of the controller. This part describes the simplest type of controller, which is the two-position, or ON-OFF, mode controller.

A two position controller is a device that has two operating conditions: completely on or completely off. Figure 16 shows the input to output, the characteristic waveform for a two position controller that switches from its "OFF" state to its "ON" state when the measured variable increases above the setpoint. Conversely, it switches from its "ON" state to its "OFF" state when the measured variable decreases below the setpoint. This device provides an output determined by whether the error signal is above or below the setpoint. The magnitude of the error signal is above or below the setpoint. The magnitude of the error signal past that point is of no concern to the controller.

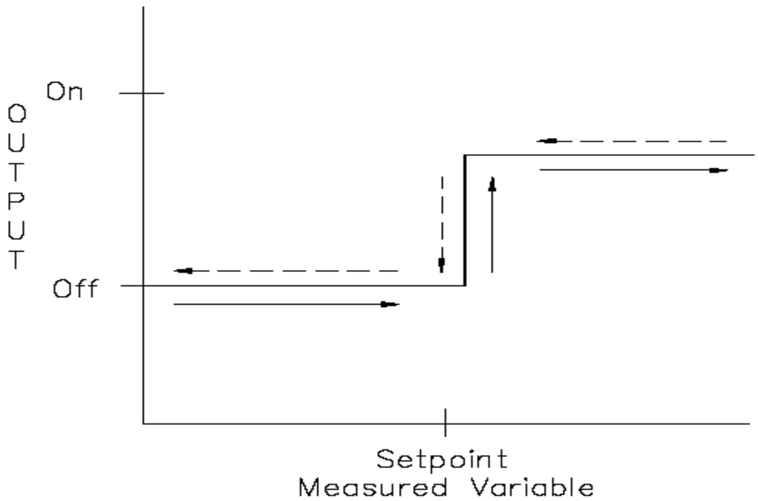


Fig. 16: Input-output relation of a two-position controller

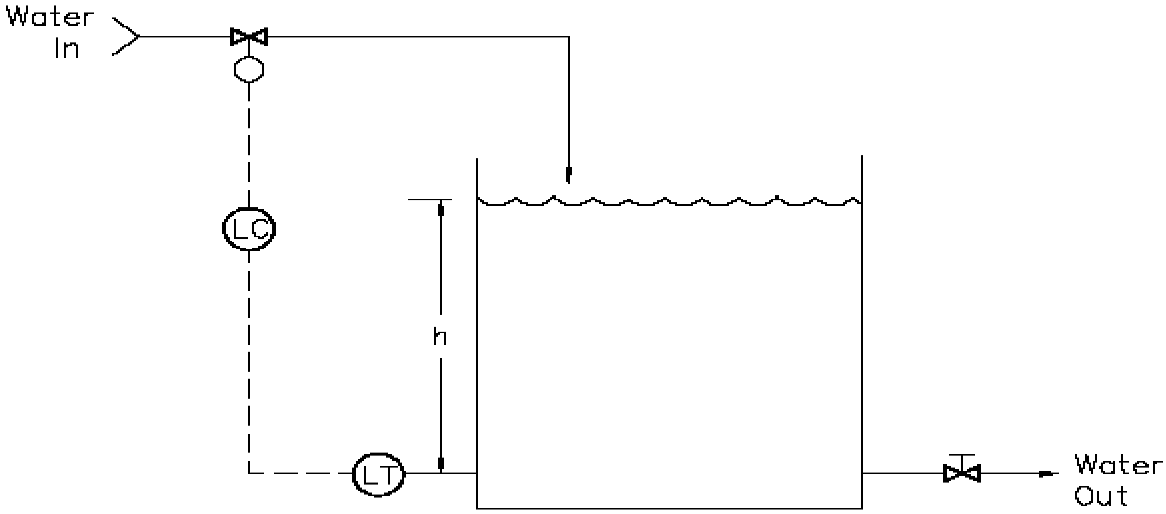


Fig. 17: Example of a two-position controller – volume of water in a tank

As an example of a system using a two-position controller is shown in Figure 17. The controlled process is the volume of water in the tank. The controlled variable is the level in the tank. It is measured by a level detector that sends information to the controller. The output of the controller is sent to the final control element, which is a solenoid valve, that controls the flow of water into the tank. As the water level decreases initially, a point is reached where the measured variable drops below the setpoint. This creates a positive error signal. The controller opens the final control element fully. Water is subsequently injected into the tank, and the water level rises. As soon as the water level rises above the setpoint, a negative error signal is developed. The negative error signal causes the controller to shut the final control element. This opening and closing of the final control element results in a cycling characteristic of the measured variable.

Valve Actuators

Actuators. By themselves, valves cannot control a process. Manual valves require an operator to position them to control a process variable. Valves that must be operated remotely and automatically require special devices to move them. These devices are called actuators. Actuators may be pneumatic, hydraulic, or electric solenoids or motors.

Pneumatic Actuators. A simplified diagram of a pneumatic actuator is shown in Figure 18. It is operated by a combination of force created by air and spring force. The actuator positions a control valve by transmitting its motion through the stem.

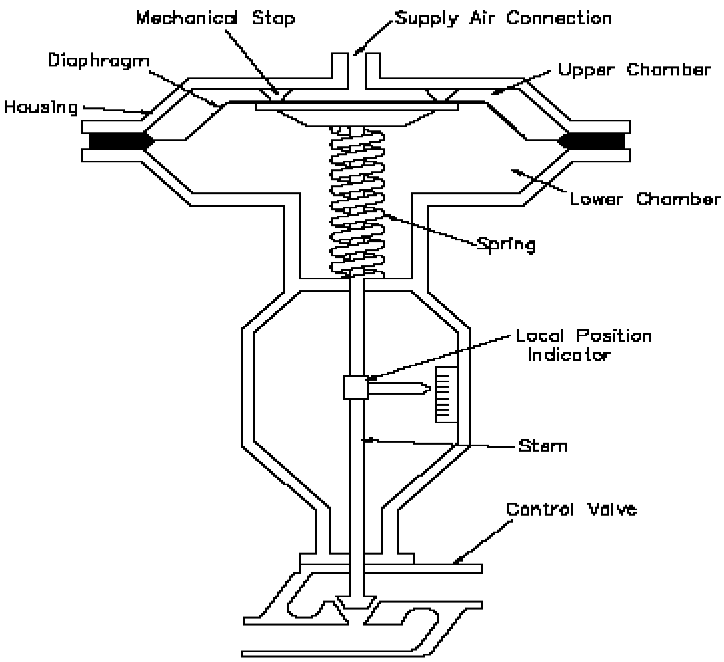


Fig. 18: Pneumatic actuator – Air to close, and spring to open

A rubber diaphragm separates the actuator housing into two air chambers. The upper chamber receives supply air through an opening in the top of the housing. The bottom chamber contains a spring that forces the diaphragm against the mechanical stops in the upper chamber. Finally, a local indicator is connected to the stem to indicate the position of the valve. The position of the valve is controlled by varying supply air pressure in the upper chamber. This results in a varying force on the top of the diaphragm. Initially, with no supply air, the spring forces the diaphragm upward against the mechanical stops and holds the valve fully open. As supply air pressure is increased from zero, its force on top of the diaphragm begins to overcome the opposing force of the spring. This causes the diaphragm to move downward and the control valve to close. With increasing supply air pressure, the diaphragm will continue to move downward and compress the spring until the control valve is fully closed. Conversely, if supply air pressure is decreased, the spring will begin to force the diaphragm upward and open the control valve. Additionally, if supply pressure is held constant at some value between zero and maximum, the valve will position at an intermediate position. Therefore, the valve can be positioned anywhere between fully open and fully closed in response to changes in supply air pressure.

A positioner is a device that regulates the supply air pressure to a pneumatic actuator. It does this by comparing the actuator's demanded position with the control valve's actual position. The demanded position is transmitted by a pneumatic or electrical control signal from a controller to the positioner. The pneumatic actuator in Figure 18 is shown in Figure 19 with a controller and positioner added.

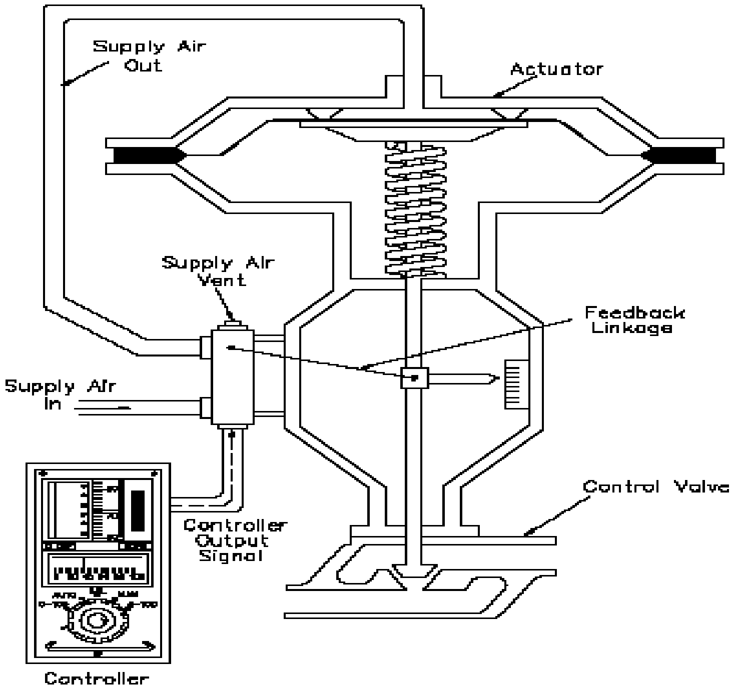


Fig. 19: Pneumatic actuator with controller and positioner

The controller generates an output signal that represents the position demanded. This signal is sent to the positioner. Externally, the positioner consists of an input connection of the control signal, a supply air input connection, a supply air output connection, a supply air vent connection, and a feedback linkage. Internally, it contains an intricate network of electrical transducers, air lines, valves, linkages, and necessary adjustments. Other positioners may also provide controls for local valve positioning and gauges to indicate supply air pressure and control air pressure (for pneumatic controllers). From an operator’s viewpoint, a description of complex internal workings of a positioner is not needed. Therefore, this discussion will be limited to inputs to and outputs from the positioner.

In Figure 19, the controller responds to a deviation of a controlled variable from set point and varies the control output signal accordingly to correct the deviation. The control output signal is sent to the positioner, which responds by increasing or decreasing the supply air to the actuator. Positioning of the actuator and control valve is fed back to the positioner through the feedback link. When the valve has reached the position demanded by the controller, the positioner stops the change in supply air pressure and holds the valve at the new position. This, in turn, corrects the controlled variable’s deviation from setpoint.

For example, as the control signal increases, a valve inside the positioner admits more supply air to the actuator. As a result, the control valve moves downward. The linkage transmits the valve position information back to the positioner. This forms a small internal feedback loop for the actuator. When the valve reaches the position that correlates to the control signal, the linkage stops the supply air flow to the actuator. This causes the actuator to stop. On the other hand, if the control signal decreases, another valve inside the positioner opens and allows the supply air pressure to decrease by venting the supply air. This causes the valve to move upward and open. When the valve has opened to the proper position, the positioner stops venting air from the actuator and stops the movement of the control valve.

An important safety feature is provided by the spring in an actuator. It can be designed to position a control valve in a safe position if a loss of supply air occurs. At a loss of supply air, the actuator in Figure 19 will fail open. This type of arrangement is referred to as "air-to-close, spring-to-open" or simply "fail-open." Some valves fail in the closed position. This type of actuator is referred to as "air-to-open, spring-to-close" or "fail-closed." This "fail-safe" concept is an important consideration in critical facility designs.

Hydraulic Actuators. Pneumatic actuators are normally used to control processes requiring quick and accurate response, as they do not require a large amount of motive force. However, when a large amount of force is required to operate a valve (for example, the main steam system valves), hydraulic actuators are normally used. Although hydraulic actuators come in many designs, piston types are most common.

A typical piston-type hydraulic actuator is shown in Figure 20. It consists of a cylinder, piston, spring, hydraulic supply and return line, and stem. The piston slides vertically inside the cylinder and separates the cylinder to two chambers. The upper chamber contains the spring and the lower chamber contains hydraulic oil.

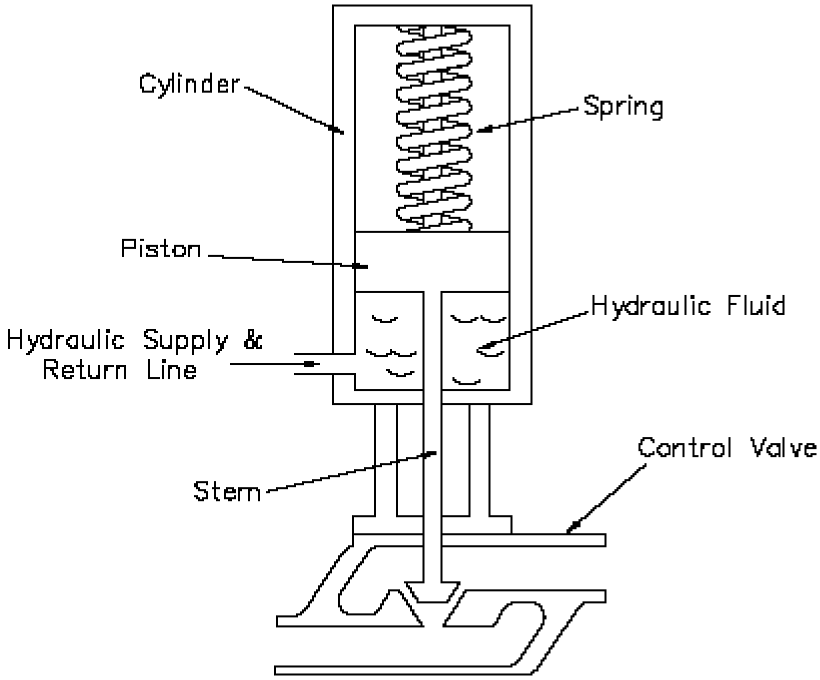


Fig. 20: Hydraulic actuator

The hydraulic supply and return line is connected to the lower chamber and allows hydraulic fluid to flow to and from the lower chamber of the actuator. The stem transmits the motion of the piston to a valve.

Initially, with no hydraulic fluid pressure, the spring force holds the valve in the closed position. As fluid enters the lower chamber, pressure in the chamber increases. This pressure results in a force on the bottom of the piston opposite to the force caused by the spring. When the hydraulic force is greater than the spring force, the piston begins to move upward, the spring compresses, and the valve begins to open. As the hydraulic pressure increases, the valve continues to open. Conversely, as hydraulic oil is drained from the cylinder, the hydraulic force becomes less than the spring force, the piston moves downward, and the valve closes. By regulating the amount of oil supplied or drained from the actuator, the valve can be positioned between fully open and fully closed.

The principles of operation of a hydraulic actuator are like those of the pneumatic actuator. Each uses some motive force to overcome the spring force to move the valve. Also, hydraulic actuators can be designed to fail-open or fail-closed to provide a fail-safe feature.

Electric Solenoid Actuators. A typical electric solenoid actuator is shown in Figure 21. It consists of a coil, armature, spring, and stem. The coil is connected to an external current supply. The spring rests on the armature to force it downward. The armature moves vertically inside the coil and transmits its motion through the stem to the valve.

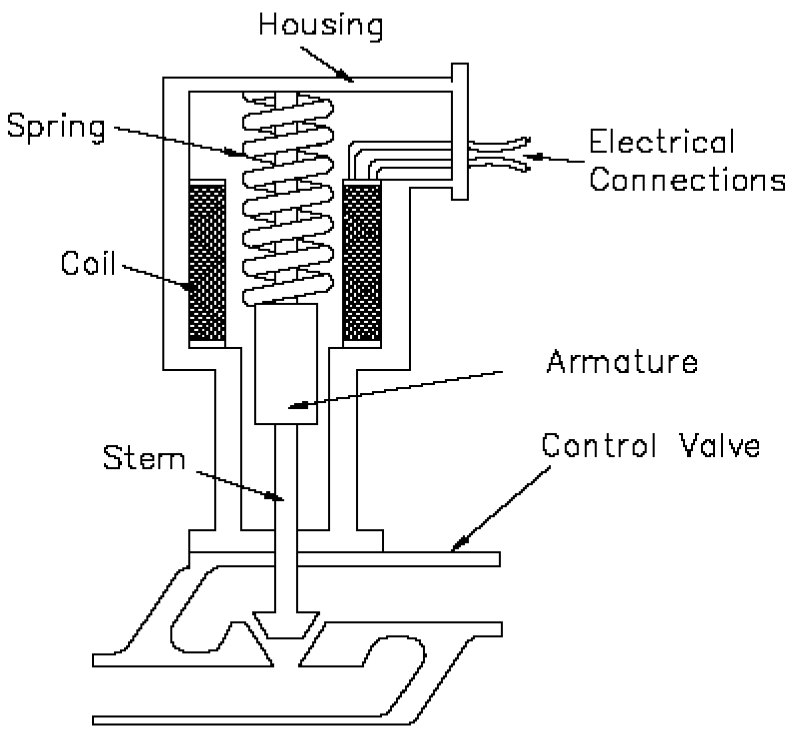


Fig. 21: Electric solenoid actuator

When current flows through the coil, a magnetic field forms around the coil. The magnetic field attracts the armature toward the center of the coil. As the armature moves upward, the spring collapses and the valve opens. When the circuit is opened and current stops flowing in the coil, the magnetic field collapses. This allows the spring to expand and shut the valve.

A major advantage of solenoid actuators is their quick operation. Also, they are much easier to install than pneumatic or hydraulic actuators. However, solenoid actuators have two disadvantages. First, they have only two positions: fully open and fully closed. Second, they don't produce much force, so they usually only operate relatively small valves.

Electric Motor Actuators. Electric motor actuators vary widely in their design and applications. Some electric motor actuators are designed to operate in only two positions (fully open or fully closed). Other electric motors can be positioned between the two positions. A typical electric motor actuator is shown in Figure 22. Its major parts include an electric motor, clutch and gear box assembly, manual handwheel, and stem connected to a valve.

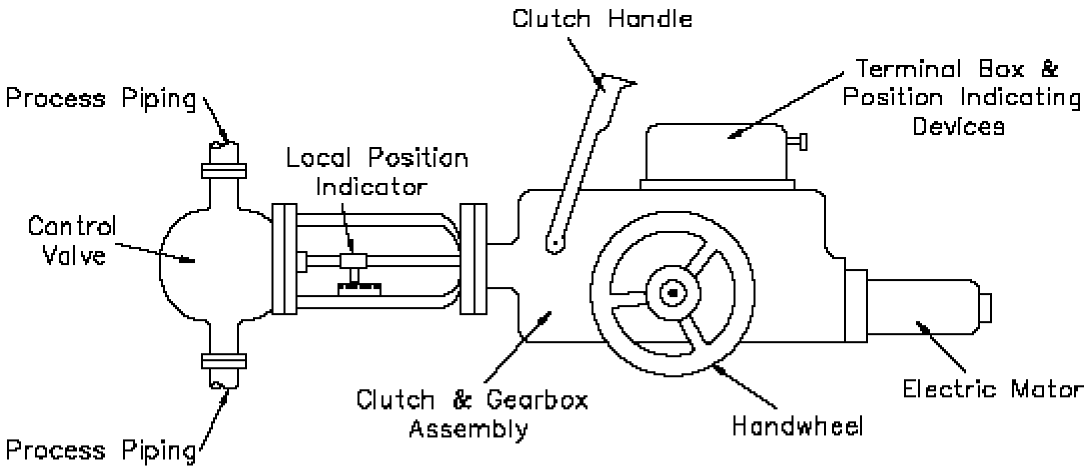


Fig. 22: Electric motor actuator

The motor moves the stem through the gear assembly. The motor reverses its rotation to either open or close the valve. The clutch and clutch lever disconnects the electric motor from the gear assembly and allows the valve to be operated manually with the handwheel. Most electric motor actuators are equipped with limit switches, torque limiters, or both. Limit switches de-energize the electric motor when the valve has reached a specific position. Torque limiters de-energize the electric motor when the amount of turning force has reached a specified value. The turning force normally is greatest when the valve reaches the fully open or fully closed position. This feature can also prevent damage to the actuator or valve if the valve binds in an intermediate position.

Design Principles of Control Systems.

General requirements of a control system:

- ✓ Any control system must be stable. This is a primary requirement.
- ✓ In addition to absolute stability, a control system must have a reasonable relative stability.
- ✓ This means fast response, with reasonable damping.
- ✓ A control system must be capable of reducing errors to zero or small tolerable value.

Note:

- The requirement of reasonable relative stability and that of steady-state accuracy tend to be incompatible.
- In designing control systems, we therefore find it necessary to make the most effective compromise between these two requirements.

Basic problems in control system design.

- In a practical control system, there are always some disturbances acting on the plant.
- The controller must take into consideration any disturbances (internal or external) which will affect the output variables.
- Performance indices must be defined to determine the optimal control signal.
- The specification of the control signal over the operating time is called the *control law*.
- The basic control problem is to determine the control law.

Design steps:

- *Analysis*. By the analysis of a control system, we mean the investigation, under specific conditions, of the performance of the system whose mathematical model is known. Since any system is made up with components, analysis must start with a mathematical description of each component. Once a mathematical model of the complete system has been derived, the manner in which analysis is carried out is independent of whether the physical system is pneumatic, electrical, mechanical … etc.
- *Design*. To design a system means to find one which accomplishes a given task. In general, the design procedure is not straightforward and will require trial-and-error method.
- *Synthesis*. By synthesis, we mean finding by a direct procedure a system that will perform in a specified way. Usually such a procedure is entirely mathematical from the start to the end of the design process. Synthesis procedures are now available for linear networks and for optimal systems.

Basic approach to control system design:

- The basic approach to the design of a practical control system will necessarily involve trial-and-error procedures.
- After the mathematical design has been completed, the control engineer simulates the model on a computer to test the behavior of the resulting system in response to various signals and disturbances.
- The system may be required to be redesigned.
- A satisfactory result will lead to the production of the prototype physical system.

Relation with subsequent chapters

The following three chapters will present the basic modeling, and theories of electric circuits. These models will be used in the later chapter for modeling control system, and studying their performances. It is important to know that some examples will method for modeling physical components for control system presentation. Some of these components are not presented in details in this course due to their advanced level; however, for control system modeling and analysis the basics of the physical components are only needed. In a later course (the mechatronics) more details about the physical structures and performances of advanced components will be given.

Chapter 2

DC Electric Circuits

Objective

- To understand the basic laws of Dc electric circuit
- To study different techniques to analysis of DC electric circuit

Introduction

A circuit consists of electrical elements connected together. Engineers use electric circuits to solve problems that are important to modem society. In particular:

1. Electric circuits are used in the generation, transmission, and consumption of electric power and energy.

2. Electric circuits are used in the encoding, decoding, storage, retrieval, transmission, and processing of information.

An electric circuit or electric network is an interconnection of electrical elements linked together in a closed path so that an electric current may flow continuously.

Consider a simple circuit consisting of two well-known electrical elements, a battery and a resistor, as shown in Fig. 1. Each element is represented by the two-terminal element shown in Fig. 2. Elements are sometimes called devices, and terminals are sometimes called nodes.

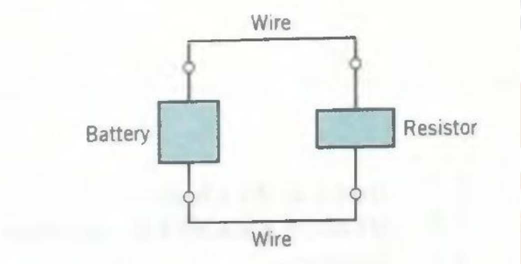


Fig. 1 A simple circuit

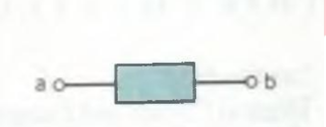


Fig. 2 A general two-terminal electrical element with terminals a and b

Charge may flow in an electric circuit. Current is the time rate of change of charge past a given point. Charge is the intrinsic property of matter responsible for electric phenomena. The current through a specified area is defined by the electric charge passing through the area per unit of time. Thus, q is defined as the charge expressed in coulombs (C).

Charge is the quantity of electricity responsible for electric phenomena. Then we can express current as

$$I = dq / dt \qquad (1)$$

The unit of current is the ampere (A); an ampere is 1 coulomb per second. Current is the time rate of flow of electric charge past a given point.

- **EX 1 :**

Find the current in an element when the charge entering the element is

$q = 12/ C$

where t is the time in seconds.

Solution

Recall that the unit of charge is coulombs, C. Then the current, from Eq. 1, is

$I = dq / dt \; = 12$ A

where the unit of current is amperes, A.

If the charge q is known, the current i is readily found using Eq. 1. Alternatively, if the current i is known, the charge q is readily calculated. Note that from Eq. 2, we obtain :

$$q = \int i \;\; d x \qquad (2)$$

The basic variables in an electrical circuit are current and voltage. These variables describe the flow of charge through the elements of a circuit and the energy required to cause charge to flow. Figure 3 shows the notation we use to describe a voltage. There are two parts to this notation: a value (perhaps represented by a variable name) and an assigned direction. The value of a voltage may be positive or negative. direction of a voltage is given by its polarities (+, -) . As a matter of vocabulary, we say that a voltage exists across an element. Figure 3 shows that there are two ways to label the voltage across an element. The voltage v_{ba} is proportional to the work required to move a positive charge from terminal a to terminal b. On the other hand, the voltage v_{ab} is proportional to the work required to move a positive charge from terminal b to terminal a. We sometimes read v_{ba} as “the voltage at terminal b with respect to terminal a.” Similarly, v_{ab} can be read as ‘the voltage at terminal a with respect to terminal b.” Alternatively, we sometimes say that vba is the voltage drop from terminal a to terminal b. The voltages v_{ab} and v_{ba} are similar but different. They have the same magnitude but different polarities. This means that

$$v_{ab} = -v_{ba} \quad (3)$$

When considering v_{ba}, terminal b is called the “+ terminal” and terminal a is called the terminal.” On the other hand, when talking about v_{ab}, terminal a is called the “ + terminal” and terminal b is called the “ V terminal.” The voltage across an element is the work (energy) required to move a unit positive charge from the “—“ terminal to the “+” terminal. The unit of voltage is the volt, V.

The equation for the voltage across the element is

$$V = dw/dq \quad (4)$$

where v is voltage, w is energy (or work), and q is charge. A charge of 1 coulomb delivers an energy of 1 joule as it moves through a voltage of 1 volt.

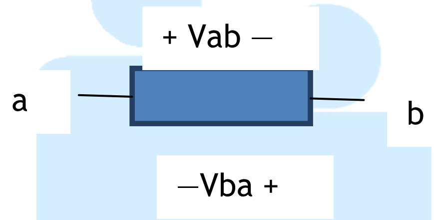


Fig. 3 Voltage across a circuit element

The power and energy delivered to an element are of great importance. For example, the useful output of an electric lightbulb can be expressed in terms of power. We know that a 300-watt bulb delivers more light than a 100-watt bulb. Power is the time rate of expending or absorbing energy. Thus, we have the equation

$$P= dw / dt \quad (5)$$

where p is power in watts, w is energy in joules, and / is time in seconds. The power associated with the charge flow through an element is

$$p= dw /dt = dw /dq * dq / dt = v * i \quad (6)$$

From Eq. 6, we see that the power is simply the product of the voltage across an element times the current through the element. The power has units of watts.

Two circuit variables are assigned to each element of a circuit: a voltage and a current. Figure 4 shows that there are two different ways to arrange the direction of the current and the polarity of the voltage. In Figure 4-a, the current enters the circuit element at the + terminal of the voltage and exits at the - terminal. In contrast, in Figure 4-b, the current enters the circuit element at the - terminal of the voltage and exits at the + terminal. First, consider Figure 4-a. When the current enters the circuit element at the + terminal of the voltage and exits at the - terminal, the voltage and current are said to “adhere to the passive convention.” In the passive convention, the voltage pushes a positive charge in the direction indicated by the current. Accordingly, the power calculated by multiplying the element voltage by the element current

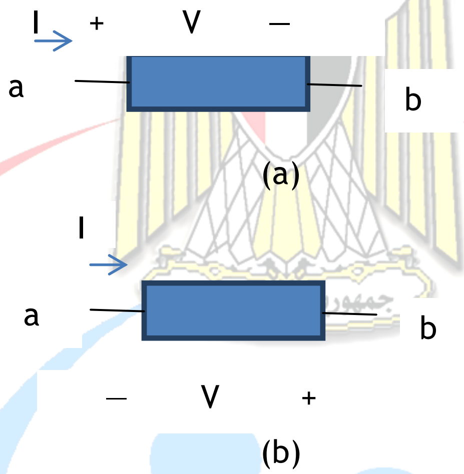


Fig. 4 (a) The passive convention is used for element voltage and current
(b) The passive convention is not used.

$$p = vi \qquad (7)$$

is the power absorbed by the element. (This power is also called “the power received by the element” and “the power dissipated by the element.”)

Next, consider Figure 4-b. Here the passive convention has not been used. Instead, the current enters the circuit element at the - terminal of the voltage and exits at the + terminal. In this case, the voltage pushes a positive charge in the direction opposite to the direction indicated by the current. Accordingly, when the element voltage and current do not adhere to the passive convention, the power calculated by multiplying the element voltage by the element current is the power supplied by the element. The power absorbed by an element and the power supplied by that same element are

related by

$$\text{power absorbed} = -\text{power supplied} \qquad (8)$$

- **Ex. 2**

Let us consider the element shown in Figure 4-a when v = 4 V and i = 10 A. Find the power absorbed by the element and the energy absorbed over a 10-s interval.

Solution

The power absorbed by the element is

p = vi = 4 • 10 = 40 W

The energy absorbed by the element is

W = $\int_0^{10} P\,dt$ = 40 * 10 = 400 J

- **<u>EX 3 :</u>**

Fig. 5 shows four circuit elements identified by the letters A, B, C and D

I. Which of the devices supply 12 W?
II. Which of the devices absorb 12 W?
III. What is the value of the power received by device B?
IV. What is the value of the power delivered by device B?
V. What is the value of the power delivered by device D?

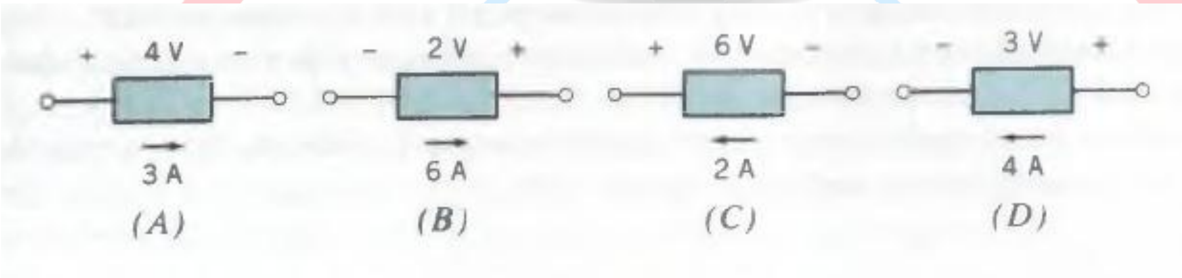


Fig. 5

Answers: (a) B and C, (b) A and D, (c) -1 2 W, (d) 12 W, (e) -12 W

In fact, the law of conservation of energy must be obeyed in any electric circuit. For this reason, the algebraic sum of power in a circuit, at any instant of time, must be zero. On other hand it can be represented as the algebraic sum of power absorbed in a circuit should equal the algebraic sum of power supplied in a circuit.

Basic Laws

If the current does not change with time, but remains constant, we call it a direct current (dc). A direct current (dc) is a current that remains constant with time. In the next parts of the chapter we will deal with DC circuits.

Circuit Elements

An element is the basic building block of a circuit. An electric circuit is simply an interconnection of the elements. Circuit analysis is the process of determining voltages across (or the currents through) the elements of the circuit. There are two types of elements found in electric circuits: passive elements and active elements. An active element is capable of generating energy while a passive element is not. Examples of passive elements are resistors, capacitors, and inductors. Typical active elements include generators, batteries, and operational amplifiers.

The most important active elements are voltage or current sources that generally deliver power to the circuit connected to them. There are two kinds of sources: independent and dependent sources.

An ideal independent source is an active element that provides a specified voltage or current that is completely independent of other circuit variables. In other words, an ideal independent voltage source delivers to the circuit whatever current is necessary to maintain its terminal voltage. Physical sources such as batteries and generators may be regarded as approximations to ideal voltage sources. Figure 6 shows the symbols for independent voltage sources. Notice that both symbols in Fig. 6 (a) and (b) can be used to represent a dc voltage source, but only the symbol in Fig. 6 (a) can be used for a time-varying voltage source. Similarly, an ideal independent current source is an active element that provides a specified current completely independent of the voltage across the source. That is, the current source delivers to the circuit whatever voltage is necessary to maintain the designated current. The symbol for an independent current source is displayed in Fig. 7, where the arrow indicates the direction of current i.

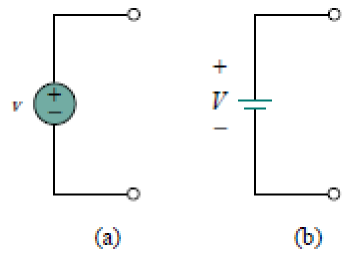


Fig. 6 symbols for independent voltage sources (a) used for constant or time-varying voltage (b) used for constant voltage (dc)

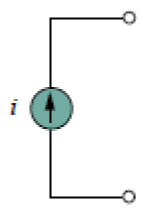


Fig. 7 symbol for independent current source

Resistance is the physical property of an element or device that impedes the flow of current; it is represented by the symbol R. where A is the cross-sectional area, ρ the resistivity, L the length, and v the voltage across the wire element. Ohm, defined the constant resistance R as

$$R = \rho L / A \quad \text{ohm} \qquad (9)$$

Fig. 8 shows the Resistor and the circuit symbol for resistance

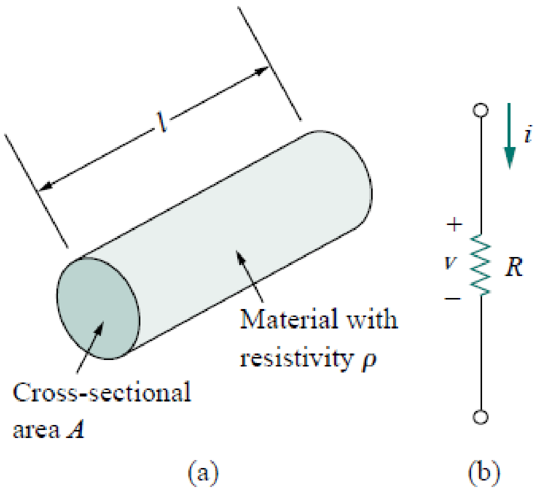


Fig. 8 (a) Resistor (b) circuit symbol for resistance

Ohm's law

Ohm's law, which related the voltage and current, was published in 1827 as :

$$V = I * R \quad \text{volt} \qquad (10)$$

Ohm's law states that the voltage v across a resistor is directly proportional to the current i flowing through the resistor.

The unit of resistance R was named the ohm in honor of Ohm and is usually abbreviated by the Ω (capital omega) symbol, where 1 Ω= 1 V/A.

When resistance value is approaching infinite it is called open circuit and when the resistance value is approaching zero is called short circuit as shown in Fig. 9

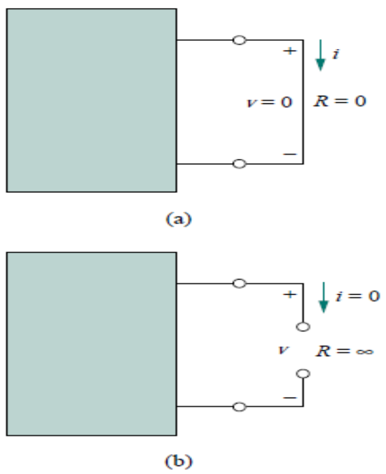


Fig. 9 (a) short circuit (R=0) (b) open circuit (R= infinite)

A useful quantity in circuit analysis is the reciprocal of resistance R, known as conductance and denoted by G. Conductance is the ability of an element to conduct electric current; it is measured in mho or siemens (S).

- **Ex 4 :**

An electric iron draws 2 A at 120 V. Find its resistance.

Solution:

From Ohm's law,

$R = v / I = 120 / 2 = 60\ \Omega$

- **Ex 5**

In the circuit shown in Fig. 10, calculate the current i, the conductance G, and the power p.

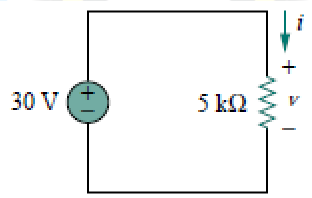


Fig. 10

Solution:

The voltage across the resistor is the same as the source voltage (30 V) because the resistor and the voltage source are connected to the same pair of terminals. Hence, the current is

$I = V / R = 30 / (5 * 10^3) = 6\ mA$

The conductance is

$G = 1 / R = 1 / (5 * 10^3) = 0.2\ mS$

We can calculate the power in various ways

$P = v * I = I^2 * R = V^2 * G = 180\ mW$

NODES, BRANCHES, AND LOOPS

Since the elements of an electric circuit can be interconnected in several ways, we need to understand some basic concepts of network topology. To differentiate between a circuit and a network, we may regard a network as an interconnection of elements or devices, whereas a circuit is a network providing one or more closed paths. The convention, when addressing network topology, is to use the word network rather than circuit. We do this even though the words network and circuit mean the same thing when used in this context. In network topology, we study the properties relating to the placement of elements in the network and the geometric configuration of the network. Such elements include branches, nodes, and loops. **A branch** represents a single element such as a voltage source or a resistor.

In other-words, a branch represents any two-terminal element. The circuit in Fig. 11 has five branches, namely, the 10-V voltage source, the 2-A current source, and the three resistors.

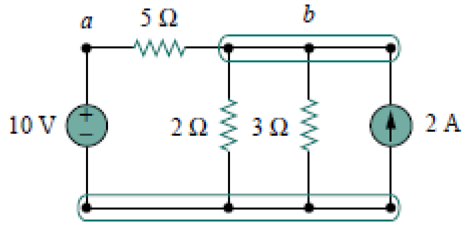


Fig. 11 Nodes, branches and loops

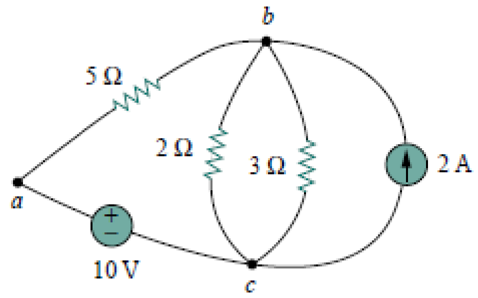


Fig. 12 The three nodes circuit of Fig. 11 is redrawn

A node is the point of connection between two or more branches.

A node is usually indicated by a dot in a circuit. If a short circuit (a connecting wire) connects two nodes, the two nodes constitute a single node. The circuit in Fig. 11 has three nodes a, b, and c. Notice that the three points that form node b are connected by perfectly conducting wires and therefore constitute a single point. The same is true of the four points forming node c. We demonstrate that the circuit in Fig. 11 has only three nodes by redrawing the circuit in Fig. 12. The two circuits in Figs. 11 and 12 are identical. However, for the sake of clarity, nodes b and c are spread out with perfect conductors as in Fig. 11.**A loop** is any closed path in a circuit. A loop is a closed path formed by starting at a node, passing through a set of nodes, and returning to the starting node without passing through any node more than once. A loop is said to be independent if it contains abranch which is not in any other loop.

Independent loops or paths result in independent sets of equations. For example, the closed path abca containing the 2 Ω resistor in Fig. 12 is a loop. Another loop is the closed path bcb containing the 3 Ω resistor and the current source. Although one can identify six loops in Fig. 12, only three of them are independent.

As the next two definitions show, circuit topology is of great value to the study of voltages and currents in an electric circuit. Two or more elements are **in series** if they are cascaded or connected sequentially and consequently carry the same current. Two or more elements are **in parallel** if they are connected to the same two nodes and consequently have the same voltage across them.

Elements are in series when they are chain-connected or connected sequentially, end to end. For example, two elements are in series if they share one common node and no other element is connected to that common node. Elements in parallel are connected to the same pair of terminals. Elements may be connected in a way that they are neither in series nor in parallel. In the circuit shown in Fig. 11, the voltage source and the 5 Ω resistor are in series because the same current will flow through them. The 2 Ω resistor, the 3 Ω resistor, and the current source are in parallel because they are connected to the same two nodes (b and c) and consequently have the same voltage across them. The 5 Ω and 2 Ω resistors are neither in series nor in parallel with each other.

KIRCHHOFF'S LAWS

Ohm's law by itself is not sufficient to analyze circuits. However, when it is coupled with Kirchhoff's two laws, we have a sufficient, powerful set of tools for analyzing a large variety of electric circuits. Kirchhoff's laws were first introduced in 1847 by the German physicist Gustav Robert Kirchhoff (1824–1887). These laws are formally known as Kirchhoff's current law (KCL) and Kirchhoff's voltage law (KVL). Kirchhoff's first law is based on the law of conservation of charge, which requires that the algebraic sum of charges within a system cannot change. Kirchhoff's current law (KCL) states that the algebraic sum of currents entering a node (or a closed boundary) is zero. Mathematically, KCL implies that

$$\sum_{n=1}^{N} in = 0$$

where N is the number of branches connected to the node and i_n is the n^{th} current entering (or leaving) the node. By this law, currents entering a node may be regarded as positive, while currents leaving the node may be taken as negative or vice versa. Consider the node in Fig. 13. Applying KCL gives

$$i_1 + (-i_2) + i_3 + i_4 + (-i_5) = 0 \qquad (11)$$

since currents i_1, i_3, and i_4 are entering the node, while currents i_2 and i_5 are leaving it. By rearranging the terms, we get

$$i_1 + i_3 + i_4 = i_2 + i_5 \qquad (12)$$

Equation (12) is an alternative form of KCL:

The sum of the currents entering a node is equal to the sum of the currents leaving the node.

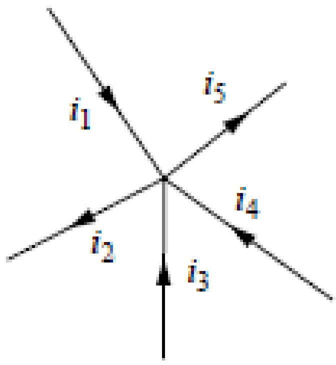


Fig. 13 currents at a node illustrating KCL

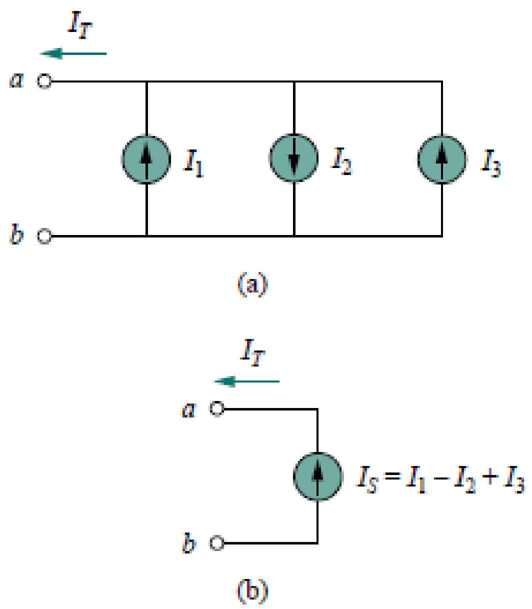


Fig. 14 Current sources in parallel (a) original circuit (b) equivalent circuit

A simple application of KCL is combining current sources in parallel. The combined current is the algebraic sum of the current supplied by the individual sources. For example, the current sources shown in Fig. 14 (a) can be combined as in Fig. 14 (b). The combined or equivalent current source can be found by applying KCL to node a.

$$I_T + I_2 = I_1 + I_3$$

$I_T = I_1 - I_2 + I_3$ (13)

A circuit cannot contain two different currents, I_1 and I_2, in series, unless $I_1 = I_2$; otherwise KCL will be violated. Kirchhoff's second law is based on the principle of conservation of energy:

Kirchhoff's voltage law (KVL) states that the algebraic sum of all voltages around a closed path(or loop) is zero. Expressed mathematically, KVL states that

$$\sum_{m=1}^{M} Vm = 0$$

Where m is the number of voltages in the loop (or the number of branches in the loop) and Vm is the m_{th} voltage. To illustrate KVL, consider the circuit in Fig. 15. The sign on each voltage is the polarity of the terminal encountered first as we travel around the loop. We can start with any branch and go around the loop either clockwise or counterclockwise. Suppose we start with the voltage source and go clockwise around the loop as shown; then voltages would be $-V_1, +V_2, +V_3, -V_4$, and $+V_5$, in that order. For example, as we reach branch 3, the positive terminal is met first; hence we have $+v_3$. For branch 4, we reach the negative terminal first; hence, $-V_4$. Thus, KVL yields

$-V_1 + V_2 + V_3 - V_4 + V_5 = 0$ (14)

Rearranging terms gives

$V_2 + V_3 + V_5 = V_1 + V_4$

Which may be interpreted as:

Sum of voltage drops = Sum of voltage rises (15)

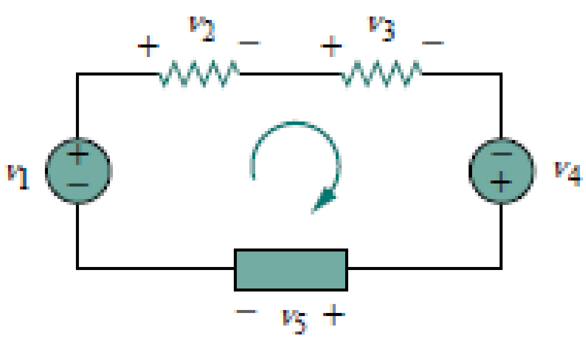


Fig. 15 single loop circuit

This is an alternative form of KVL. Notice that if we had traveled counterclockwise, the result would have been $+v_1$, $-v_5$, $+v_4$, $-v_3$, and $-v_2$, which is the same as before except that the signs are reversed. Hence, Eqs. (14) and (15) remain the same. When voltage sources are connected in series, KVL can be applied to obtain the total voltage. The combined voltage is the algebraic sum of the voltages of the individual sources. For example, for the voltage sources shown in Fig. 16 (a), the combined or equivalent voltage source in Fig. 16 (b) is obtained by applying KVL.

$$-V_{ab} + V_1 + V_2 - V_3 = 0$$

or

$$V_{ab} = V_1 + V_2 - V_3 \qquad (16)$$

To avoid violating KVL, a circuit cannot contain two different voltages V_1 and V_2 in parallel unless $V_1 = V_2$.

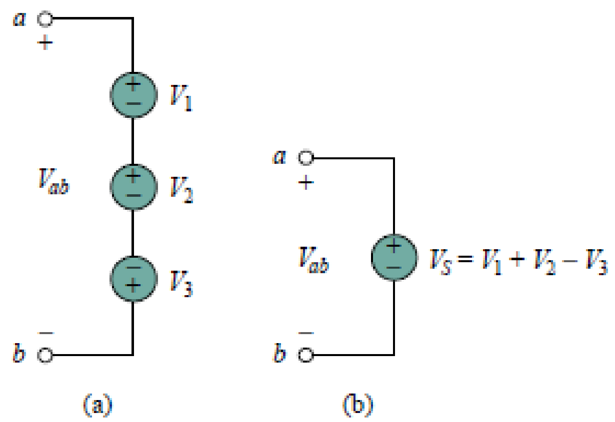


Fig. 16 Voltage sources in series (a) original circuit (b) equivalent circuit

- EX

For the circuit in Fig. 17 (a), find voltages v_1 and v_2.

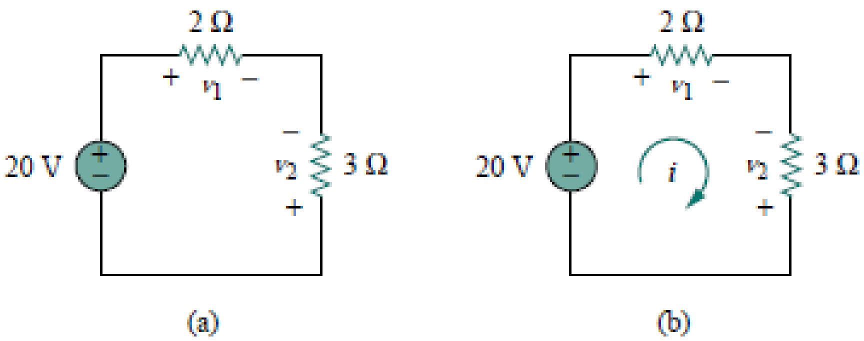


Fig. 17

Solution:

To find V_1 and V_2, we apply Ohm's law and Kirchhoff's voltage law. Assume that current "i" flows through the loop as shown in Fig. 17 (b). From Ohm's law,

$V_1 = 2i,\ V_2 = -3i$ (17)

Applying KVL around the loop gives

$-20 + V_1 - V_2 = 0$ (18)

Substituting Eq. (17) into Eq. (18), we obtain

$-20 + 2i + 3i = 0$ or $5i = 20 \Rightarrow i = 4\ A$

Substituting i in Eq. (17) finally gives

$V_1 = 8\ V,\ V_2 = -12\ V$

- Ex

Find v_1 and v_2 in the circuit of Fig. 18.

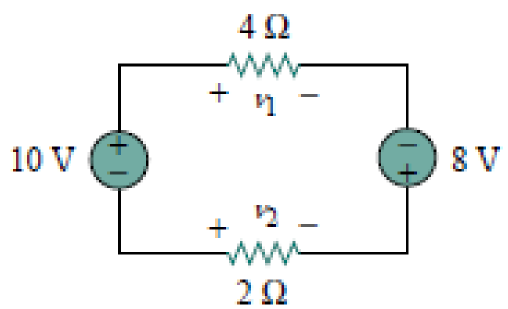


Fig. 18

Answer: 12 V, −6 V.

- Ex: Find the currents and voltages in the circuit shown in Fig. 19 (a).

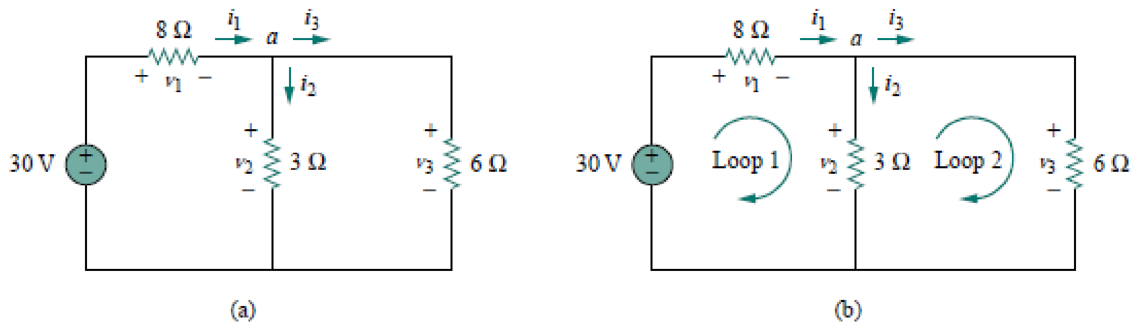


Fig. 19

Solution:

We apply Ohm's law and Kirchhoff's laws. By Ohm's law,

$v_1 = 8i_1,\ v_2 = 3i_2,\ v_3 = 6i_3$ (19)

Since the voltage and current of each resistor are related by Ohm's law as shown, we are really looking for three things: (v_1, v_2, v_3) or (i_1, i_2, i_3). At node a, KCL gives

$i_1 - i_2 - i_3 = 0$ (20)

Applying KVL to loop 1 as in Fig. 19 (b),

$-30 + v_1 + v_2 = 0$

We express this in terms of i_1 and i_2 as in Eq. (19) to obtain

$-30 + 8i_1 + 3i_2 = 0$

or

$i_1 = (30 - 3i_2) / 8$ (21)

Applying KVL to loop 2,

$-v_2 + v_3 = 0 \quad \Rightarrow v_3 = v_2$ (22)

as expected since the two resistors are in parallel. We express v_1 and v_2 in terms of i_1 and i_2 as in Eq. (19). Equation (22) becomes

$6i_3 = 3i_2 \quad \Rightarrow \quad i_3 = i_2 / 2$ (23)

Substituting Eqs. (21) and (23) into (20) gives

$(30 - 3i_2)/8 - i_2 - i_2/2 = 0$ or $i_2 = 2$ A.

From the value of i_2, we now use Eqs. (19) to (23) to obtain

$i_1 = 3$ A, $i_3 = 1$ A, $v_1 = 24$ V, $v_2 = 6$ V, $v_3 = 6$ V

- Ex

Find the currents and voltages in the circuit shown in Fig. 20.

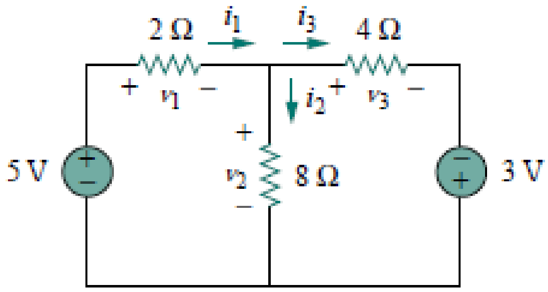


Fig. 20

Answer: v_1 = 3 V, v_2 = 2 V, v_3 = 5 V, i_1 = 1.5 A, i_2 = 0.25 A, i_3 =1.25 A.

Series and Parallel Resistors

The need to combine resistors in series or in parallel occurs so frequently that it warrants special attention. The process of combining the resistors is facilitated by combining two of them at a time. With this in mind, consider the single-loop circuit of Fig. 21. The two resistors are in series, since the same current i flows in both of them. Applying Ohm's law to each of the resistors, we obtain

$$v_1 = iR_1,\ v_2 = iR_2 \tag{24}$$

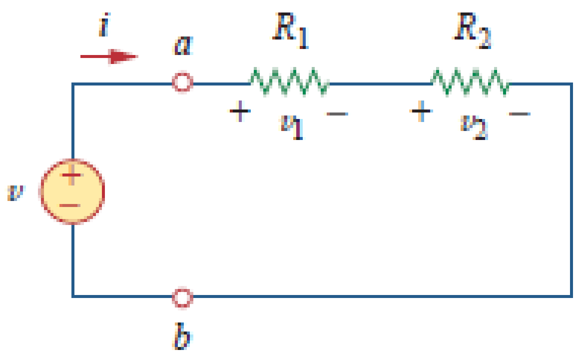


Fig. 21 A single loop circuit with two resistors in series

If we apply KVL to the loop (moving in the clockwise direction), we have

$$-v + v_1 + v_2 = 0 \tag{25}$$

Combining Eqs. (24) and (25), we get

$$v = v_1 + v_2 = i(R_1 + R_2) \tag{26}$$

or

$i = V/(R_1 + R_2)$ (27)

Notice that Eq. (26) can be written as

$V = iR_{eq}$ (28)

Implying that the two resistors can be replaced by an equivalent resistor; that is,

$R_{eq} = R_1 + R_2$ (29)

Thus, Fig. 21 can be replaced by the equivalent circuit in Fig. 22. The two circuits in Figs. 21 and 22 are equivalent because they exhibit the same voltage-current relationships at the terminals a-b. An equivalent circuit such as the one in Fig. 22 is useful in simplifying the analysis of a circuit. In general, the equivalent resistance of any number of resistors connected in series is the sum of the individual resistances. For N resistors in series then,

$R_{eq} = R_1 + R_2 + \ldots\ldots\ldots\ldots\ldots\ldots + R_N$ (30)

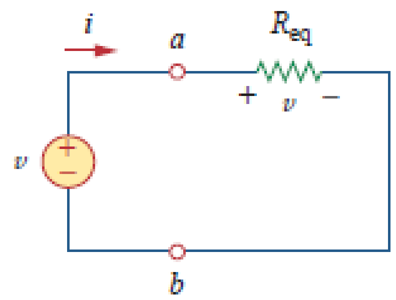


Fig. 22 Equivalent circuit

To determine the voltage across each resistor in Fig. 21, we substitute

Eq. (26) into Eq. (24) and obtain

$V_1 = [V R_1 / (R_1 + R_2)]$, $V_2 = [V R_2 / (R_1 + R_2)]$ (31)

Notice that the source voltage V is divided among the resistors in direct proportion to their resistances; the larger the resistance, the larger the voltage drop. This is called the principle of voltage division, and the circuit in Fig. 21 is called a voltage divider. In general, if a voltage divider has N resistors in series with the source voltage V, (R_1, R_2,..........R_N) , the n_{th} resistor (R_n) will have a voltage drop of

$V_n = V R_n / (R_1 + R_2 + \ldots\ldots + R_N)$ (32)

Parallel Resistors

Consider the circuit in Fig. 23, where two resistors are connected in parallel and therefore have the same voltage across them. From Ohm's law,

$V = i_1R_1 = i_2R_2$

$i_1 = V / R_1,\ i_2 = V/R_2$ (33)

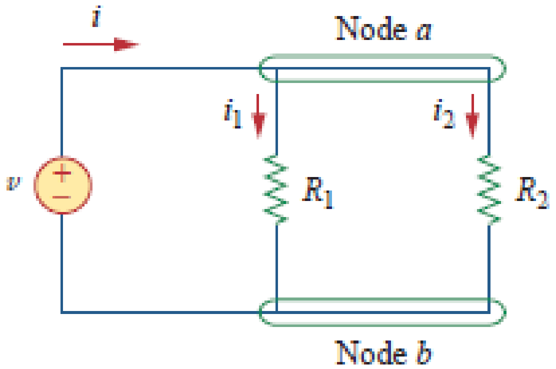


Fig. 23 Two resistor in parallel

Applying KCL at node a gives the total current i as

$i = i_1 + i_2$ (34)

Substituting Eq. (33) into Eq. (34), we get

$i = V/R_1 + V/R_2 = V\ (1/R_1 + 1R_2) = V / R_{eq}$ (35)

Where R_{eq} is the equivalent resistance of the resistors in parallel:

$1 / R_{eq} = (1/R_1 + 1/R_2)$ (36)

We can extend the result in Eq. (36) to the general case of a circuit with N resistors in parallel. The equivalent resistance is

$1 / R_{eq} = (1/R_1 + 1/R_2 + \ldots\ldots\ldots\ldots. + 1/R_N)$ (37)

This means that we may replace the circuit in Fig. 23 with that in Fig. 24.

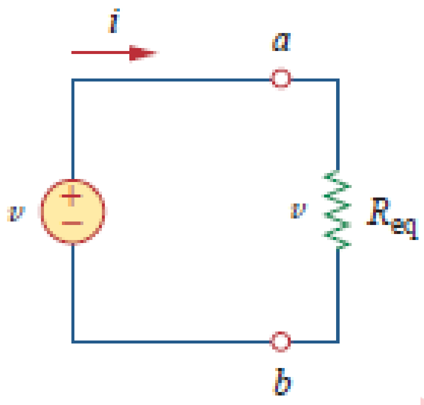


Fig. 24 Equivalent circuit

Given the total current i entering node a in Fig. 23, how do we obtain current i_1 and i_2 ? and We know that the equivalent resistor has the same voltage, or

$$V = i\,R_{eq} = \frac{i\,R_1 R_2}{R_1+R_2} \qquad (38)$$

Thus

$$i_1 = \frac{i\,R_2}{R_1+R_2}, \quad i_2 = \frac{i\,R_1}{R_1+R_2} \qquad (39)$$

Which shows that the total current i is shared by the resistors in inverse proportion to their resistances. This is known as the principle of current division, and the circuit in Fig. 23 is known as a current divider. Notice that the larger current flows through the smaller resistance.

Ex: Find R_{eq} for the circuit in Figure 25

Solution

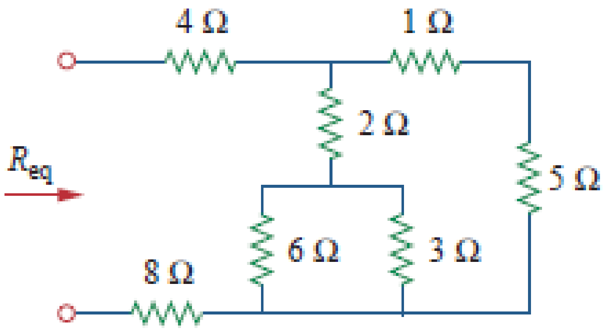


Fig. 25

To get R_{eq} the 6 Ω and 3 Ω are in parallel

$$6\ \Omega \parallel 3\Omega = \frac{6 \times 3}{6 + 3} = 2\ \Omega$$

Also 1 Ω and 5 Ω are in series, hence their equivalent resistance is

$$1\ \Omega + 5\ \Omega = 6\ \Omega$$

Thus circuit in Figure 25 is reduced to Figure 26 (a),

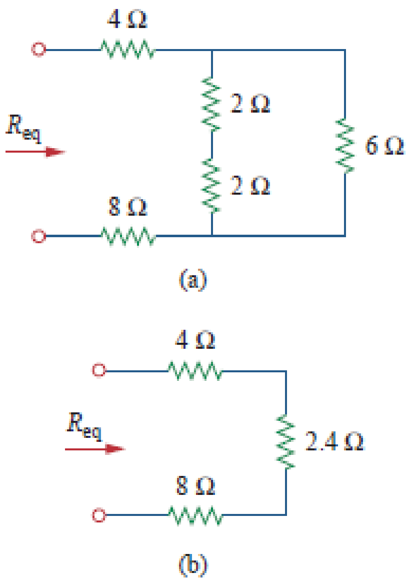


Fig. 26

We notice that the two 2 Ω resistors are in series so the equivalent resistance is

$$2\ \Omega + 2\ \Omega = 4\ \Omega$$

This 4 Ω resistor is in parallel with the 6 Ω in Fig. 26 (a) their equivalent resistance is

$$4\ \Omega \parallel 6\ \Omega = \frac{4 \times 6}{4 + 6} = 2.4\ \Omega$$

The circuit in Fig. 26 (a) is replaced with that in Fig. 26 (b). Hence the equivalent resistance is

$$R_{eq} = 4\ \Omega + 2.4\ \Omega + 8\ \Omega = 14.4\ \Omega$$

- EX : By combining the resistors in Fig. 27, find R_{eq}

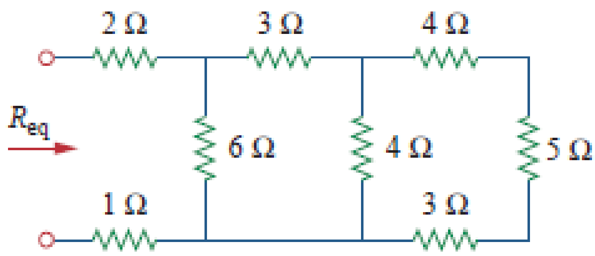


Fig. 27

Answer: 6 Ω

Ex: Calculate the equivalent resistance R_{ab} in the circuit in Figure 28

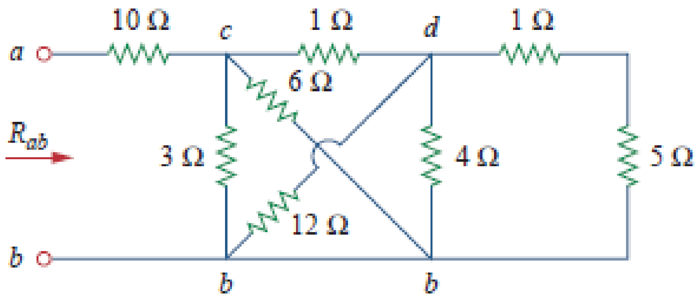


Fig. 28

The 6 Ω and 3 Ω are in parallel. Their combined resistance is

$$3\,\Omega \,\|\, 6\,\Omega = \frac{3 \times 6}{3 + 6} = 2\,\Omega$$

Similarly, The 12 Ω and 4 Ω are in parallel. Hence

$$12\,\Omega \,\|\, 4\,\Omega = \frac{12 \times 4}{12 + 4} = 3\,\Omega$$

Also The 1 Ω and 5 Ω are in series. Their equivalent resistance is

$$1\,\Omega + 5\,\Omega = 6\,\Omega$$

Thus we can replace the circuit in Figure 28 by that in Figure 29 (a). In Figure 29 (a), the 6 Ω and 2 Ω are in parallel gives 2 Ω. This 2 Ω equivalent is series with 1 Ω to give combined resistance of 3 Ω. Thus we replace circuit in Figure 29 (a) by that in Figure 29 (b). In Figure 29 (b)we combine the 2 Ω and 3 Ω are in parallel to get

$$2\,\Omega \,\|\, 3\,\Omega = \frac{2 \times 3}{2 + 3} = 1.2\,\Omega$$

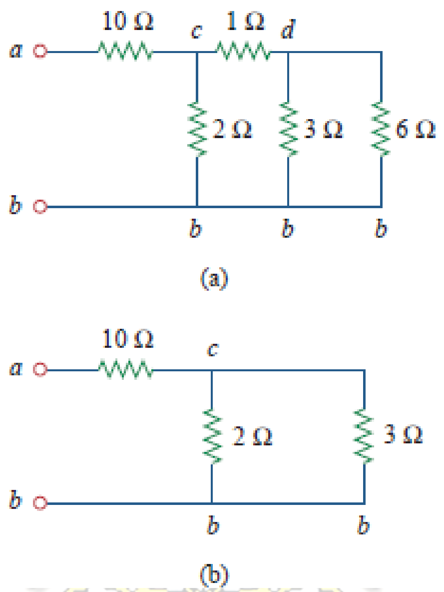


Fig. 29

This 1.2 Ω resistor is in series with the 10 Ω resistor, so that

$$R_{ab} = 10 + 1.2 = 11.2\ \Omega$$

Ex: Find i_o and V_o in the circuit shown in Figure 30 (a). calculate the power dissipated in the 3 Ω resistor.

Solution

The 6 Ω and 3 Ω are in parallel. Their combined resistance is

$$3\ \Omega \parallel 6\ \Omega = \frac{3 \times 6}{3 + 6} = 2\ \Omega$$

Thus our circuit reduces to that in Figure 30 (b). we can get Vo by ohm's law :

$$i = \frac{12}{4 + 2} = 2\ \text{A}$$

And hence $V_o = 2i = 4$ V. But $V_o = 3\ i_0 = 4$ V then $i_0 = 4/3$ A

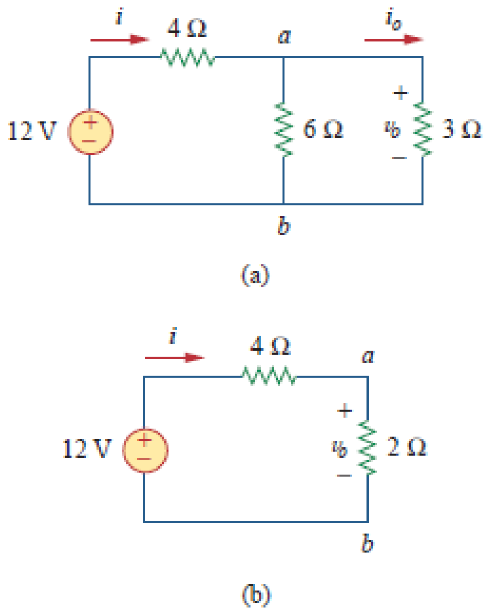


Fig. 30

The power dissipated in the 3 Ω resistor is

$P_o = V_o\ i_o = 5.333$ W

Nodal Analysis

Nodal analysis provides a general procedure for analyzing circuits using node voltages as the circuit variables. Choosing node voltages instead of element voltages as circuit variables is convenient and reduces the number of equations one must solve simultaneously. To simplify matters, we shall assume in this section that circuits do not contain voltage sources. Circuits that contain voltage sources will be analyzed in the next section. In nodal analysis, we are interested in finding the node voltages. Given a circuit with n nodes without voltage sources, the nodal analysis of the circuit involves taking the following three steps.

Steps to Determine Node Voltages:

1. Select a node as the reference node. Assign voltages V_1, V_2, …., V_n to the remaining n-1 nodes. The voltages are referenced with respect to the reference node.

2. Apply KCL to each of the n-1 non-reference nodes. Use Ohm's law to express the branch currents in terms of node voltages.

3. Solve the resulting simultaneous.

We shall now explain and apply these three steps. The first step in nodal analysis is selecting a node as the reference or datum node. The reference node is commonly called the ground since it is assumed to have zero potential. A reference node is indicated by any of the three symbols in Fig. 31. The type of ground in Fig. 31 (c) is called a chassis ground and is used in devices where the case, enclosure, or chassis acts as a reference point for all circuits. When the potential of the earth is used as reference, we use the earth ground in Fig. 31 (a) or (b). We shall always use the symbol in Fig. 31 (b). Once we have selected a reference node, we assign voltage designations to non-reference nodes. Consider, for example, the circuit in Fig. 32 (a). Node 0 is the reference node (V=0). While nodes 1 and 2 are assigned voltages v_1 and v_2 respectively. Keep in mind that the node voltages are defined with respect to the reference node. As illustrated in Fig. 32 (a), each node voltage is the voltage rise from the reference node to the corresponding non-reference node or simply the voltage of that node with respect to the reference node as the second step, we apply KCL to each non-reference node in the circuit. To avoid putting too much information on the same circuit, the circuit in Fig. 32 (a) is redrawn in Fig. 32 (b), where we now add i_1, i_2 and i_3 as the currents through resistors R_1, R_2 and R_3 respectively. At node 1, applying KCL gives

$$I_1 = I_2 + i_1 + i_2 \quad (40)$$

At Node 2,

$$I_2 + i_2 = i_3 \quad (41)$$

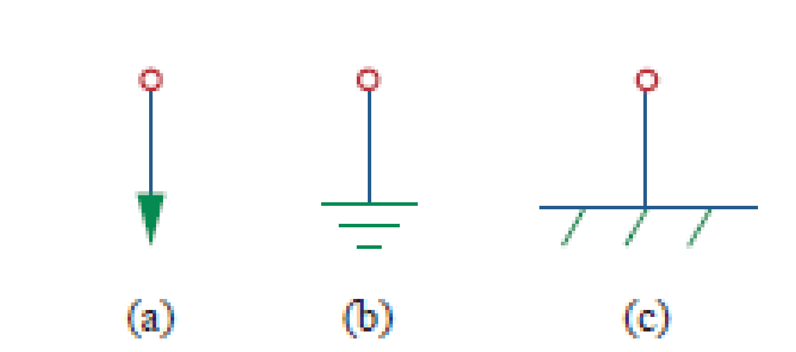


Fig. 31 Common symbols for reference node (a) common ground (b) ground (c) chassis ground

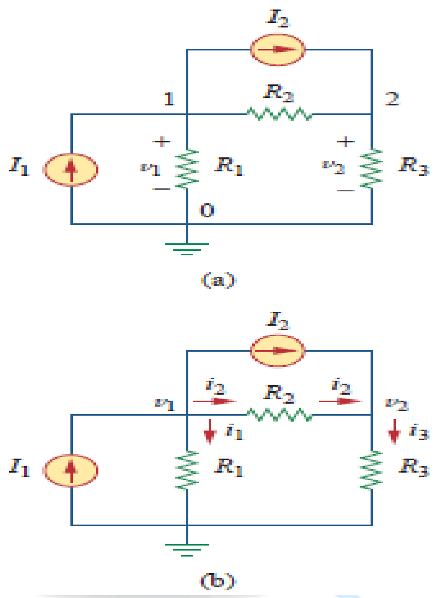


Fig. 32 Nodal analysis circuit

We now apply Ohm's law to express the unknown currents i_1, i_2 and i_3 in terms of node voltages. The key idea to bear in mind is that, since resistance is a passive element, by the passive sign convention, current must always flow from a higher potential to a lower potential. Current flows from a higher potential to a lower potential in a resistor. We can express this principle as

$$\boxed{i = \frac{v_{\text{higher}} - v_{\text{lower}}}{R}}$$

We obtain from Fig. 32(b),

$$i_1 = \frac{v_1 - 0}{R_1} \quad \text{or} \quad i_1 = G_1 v_1$$

$$i_2 = \frac{v_1 - v_2}{R_2} \quad \text{or} \quad i_2 = G_2(v_1 - v_2)$$

$$i_3 = \frac{v_2 - 0}{R_3} \quad \text{or} \quad i_3 = G_3 v_2$$

Then, by substitution in Eqs. 40 and 41

$$I_1 = I_2 + \frac{v_1}{R_1} + \frac{v_1 - v_2}{R_2}$$

$$I_2 + \frac{v_1 - v_2}{R_2} = \frac{v_2}{R_3}$$

The third step in nodal analysis is to solve for the node voltages. If we apply KCL to n-1 non-reference nodes, we obtain n-1 simultaneous equations. We solve Eqs. to obtain the node voltages v_1 and v_2 using any standard method, such as the substitution method, the elimination method, Cramer's rule, or matrix inversion. To use either of the last two methods, one must cast the simultaneous equations in matrix form.

Example: Calculate the node voltages in the circuit shown in Figure 33 (a)

Solution

Consider Figure 33 (b) where the circuit in Figure 33 (a) has been prepared for nodal analysis .

At Node 1, Applying KCL and ohm's law gives

$$i_1 = i_2 + i_3 \quad \Rightarrow \quad 5 = \frac{v_1 - v_2}{4} + \frac{v_1 - 0}{2}$$

Multiplying each term by 4

$20 = V_1 - V_2 + 2\,V_1$

Or

$3\,V_1 - V_2 = 20$ (42)

At Node 2, Applying KCL and ohm's law gives

$$i_2 + i_4 = i_1 + i_5 \quad \Rightarrow \quad \frac{v_1 - v_2}{4} + 10 = 5 + \frac{v_2 - 0}{6}$$

Multiplying each term by 12 results in

$60 + 2\,V_2 = 3V_1 - 3V_2 + 120$

Or

$-3\,V_1 + 5\,V_2 = 60$ (43)

By solving equations 42 and 43

$V_1 = 40/3 = 13.333$ V

$V_2 = 20$ V

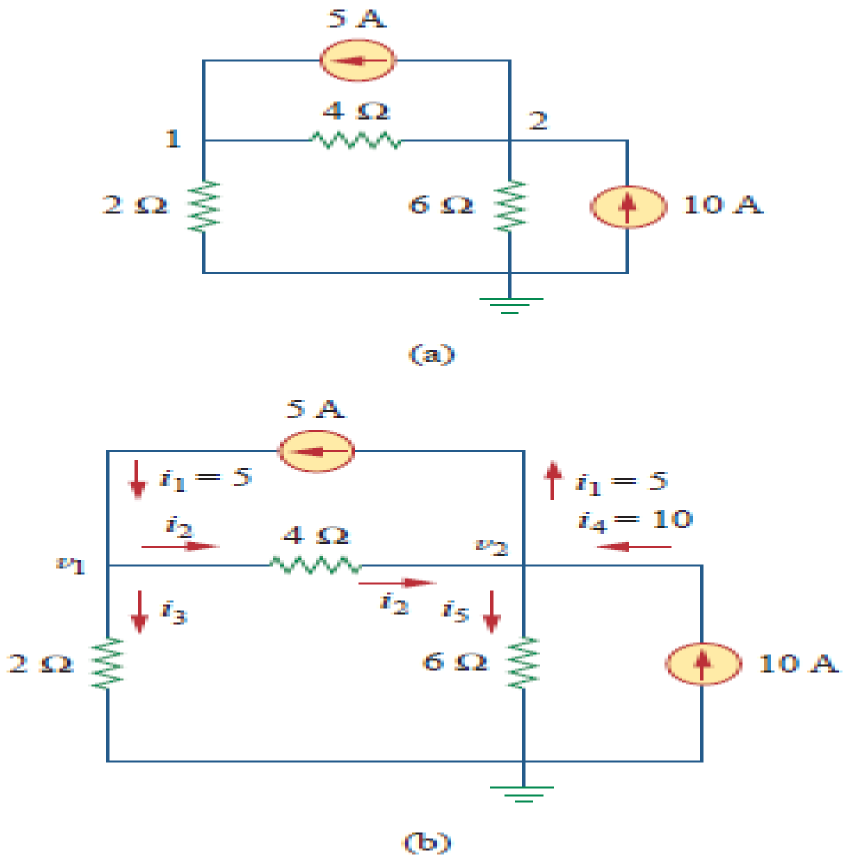


Fig. 33

Example: Obtain the node voltages in the circuit of Fig. 34.

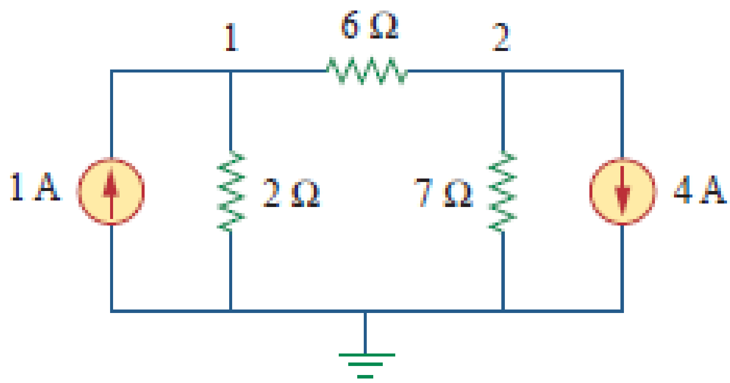


Fig. 34

Answer: v_1 =-2 V, v_2 =-14 V.

Nodal Analysis with Voltage Sources

We now consider how voltage sources affect nodal analysis. We use the circuit in Fig. 35 for illustration. Consider the following two possibilities.

■ CASE 1 If a voltage source is connected between the reference node and a non-reference node, we simply set the voltage at the non-reference node equal to the voltage of the voltage source. In Fig. 35, for example, V1 = 10 volt Thus, our analysis is somewhat simplified by this knowledge of the voltage at this node.

■ CASE 2 If the voltage source (dependent or independent) is connected between two non-reference nodes, the two non-reference nodes

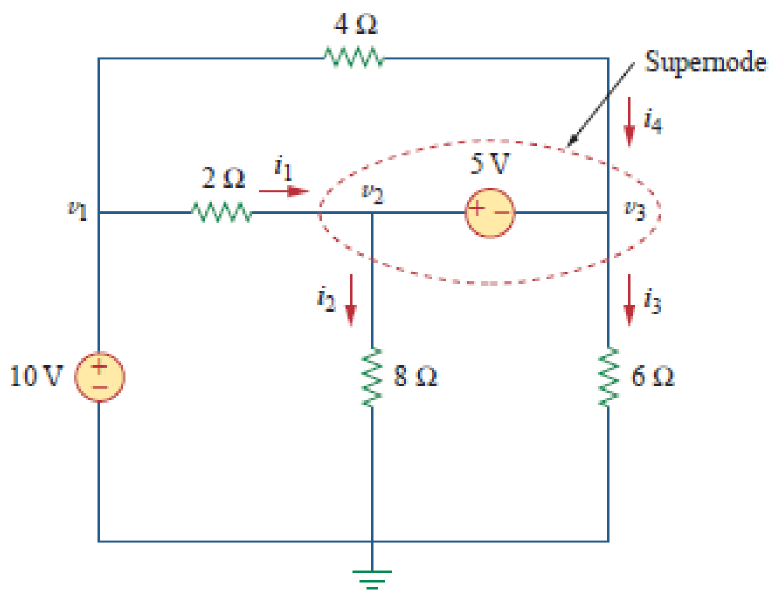


Fig. 35 supernode circuit

form a generalized node or supernode; we apply both KCL and KVL to determine the node voltages. A supernode is formed by enclosing a (dependent or independent) voltage source connected between two non-reference nodes and any elements connected in parallel with it. In Fig. 36, nodes 2 and 3 form a supernode. (We could have more than two nodes forming a single supernode.

We analyze a circuit with supernodes using the same three steps mentioned in the previous section except that the supernodes are treated differently. Why? Because an essential component of nodal analysis is applying KCL, which requires knowing the current through each element. There is no way of knowing the current through a voltage source in advance. However, KCL must be satisfied at a supernode like any other node. Hence, at the supernode in Fig. 35,

$$i_1 + i_4 = i_2 + i_3$$

$$\frac{v_1 - v_2}{2} + \frac{v_1 - v_3}{4} = \frac{v_2 - 0}{8} + \frac{v_3 - 0}{6}$$

To apply Kirchhoff's voltage law to the supernode in Fig. 35, we redraw the circuit as shown in Fig. 36. Going around the loop in the clockwise direction gives

$$-v_2 + 5 + v_3 = 0 \quad \Rightarrow \quad v_2 - v_3 = 5$$

By solving the previous equations, we obtain the node voltages.

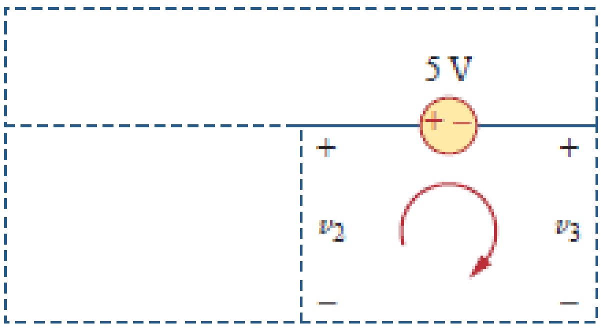


Fig. 36

Note the following properties of a supernode:

1. The voltage source inside the supernode provides a constraint equation needed to solve for the node voltages.
2. A supernode has no voltage of its own.
3. A supernode requires the application of both KCL and KVL.

Example: Find V and i in the circuit of Fig. 37.

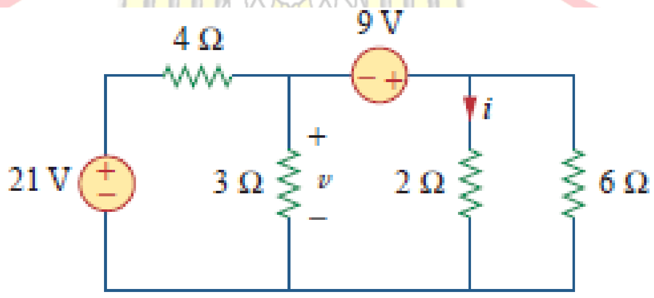


Fig. 37

Answer: - 0.6 V, 4.2 A.

Mesh Analysis

Mesh analysis provides another general procedure for analyzing circuits, using mesh currents as the circuit variables. Using mesh currents instead of element currents as circuit variables is convenient and reduces the number of equations that must be solved simultaneously. Recall that a loop is a closed path with no node passed more than once. A mesh is a loop that does not contain any other loop within it.

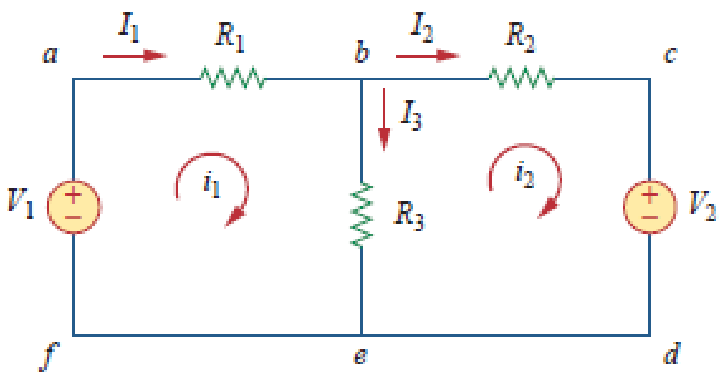


Fig. 38 A circuit with two meshes

In Fig. 38, for example, paths abefa and bcdeb are meshes, but path abcdefa is not a mesh. The current through a mesh is known as mesh current. In mesh analysis, we are

interested in applying KVL to find the mesh currents in a given circuit. In this section, we will apply mesh analysis to planar circuits that do not contain current sources. In the next section, we will consider circuits with current sources. In the mesh analysis of a circuit with n meshes, we take the following three steps. Steps to Determine Mesh Currents:

1. Assign mesh currents i_1, i_2,….. i_n to the n meshes.

2. Apply KVL to each of the n meshes. Use Ohm's law to express the voltages in terms of the mesh currents.

3. Solve the resulting n simultaneous equations to get the mesh currents.

To illustrate the steps, consider the circuit in Fig. 38. The first step requires that mesh currents i_1 and i_2 are assigned to meshes 1 and 2. Although a mesh current may be assigned to each mesh in an arbitrary direction, it is conventional to assume that each mesh current flows clockwise. As the second step, we apply KVL to each mesh. Applying KVL to mesh 1, we obtain

$$-V_1 + R_1 i_1 + R_3(i_1 - i_2) = 0$$

$$(R_1 + R_3)i_1 - R_3 i_2 = V_1 \qquad (44)$$

For mesh 2, applying KVL gives

$$R_2 i_2 + V_2 + R_3(i_2 - i_1) = 0$$

$$-R_3 i_1 + (R_2 + R_3)i_2 = -V_2 \qquad (45)$$

Note in the Eq. 44, that the coefficient of i_1 is the sum of the resistances in the first mesh, while the coefficient of i_2 is the negative of the resistance common to meshes 1 and 2. Now observe that the same is true in Eq. (45). This can serve as a shortcut way of writing the mesh equations. The third step is to solve for the mesh currents. Putting Eqs. (44) and (45) in matrix form yields

$$\begin{bmatrix} R_1 + R_3 & -R_3 \\ -R_3 & R_2 + R_3 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} V_1 \\ -V_2 \end{bmatrix}$$

Which can be solved to obtain the mesh currents i_1 and i_2 . Notice that the branch currents are different from the mesh currents unless the mesh is isolated. To distinguish between the two types of currents, we use i for a mesh current and I for a branch current. The current elements I_1, I_2 and I_3 are algebraic sums of the mesh currents. It is evident from Fig. 38 that

$$I_1 = i_1, \qquad I_2 = i_2, \qquad I_3 = i_1 - i_2$$

Ex: Calculate the mesh currents i_1 and i_2 of the circuit of Fig. 40.

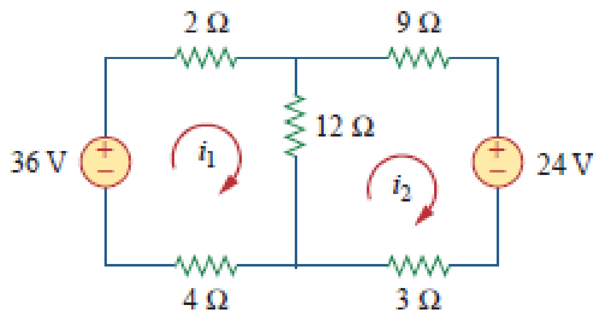


Fig. 40

Answer: i_1 = 2 A, i_2 = 0 A.

Mesh Analysis with Current Sources

Applying mesh analysis to circuits containing current sources (dependent or independent) may appear complicated. But it is actually much easier than what we encountered in the previous section, because the presence of the current sources reduces the number of equations. Consider the following two possible cases.

- CASE 1 When a current source exists only in one mesh: Consider the circuit in Fig. 41, for example. We set i_2 = -5 A and write a mesh equation for the other mesh in the usual way; that is,

$$-10 + 4i_1 + 6(i_1 - i_2) = 0 \qquad \Rightarrow \qquad i_1 = -2 \text{ A}$$

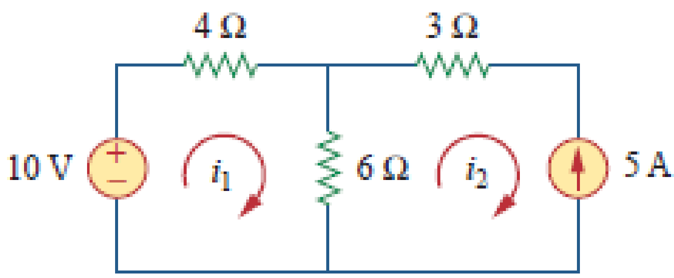


Fig. 41

- CASE 2 When a current source exists between two meshes: Consider the circuit in Fig. 42 (a), for example. We create a supermesh by excluding the current source and any elements connected in series with it, as shown in Fig. 42 (b). Thus, A supermesh results when two meshes have a (dependent or independent) current source in common.

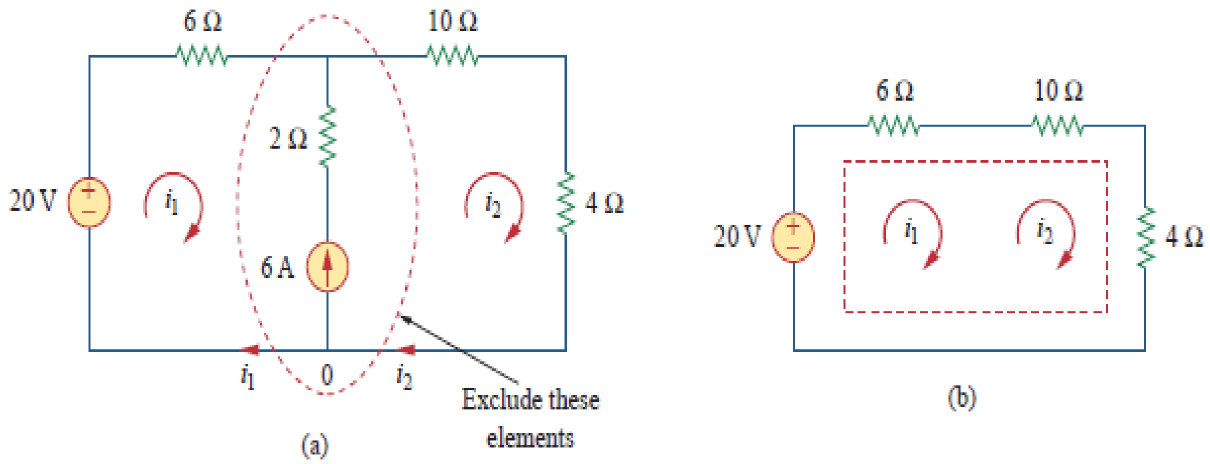


Fig. 42 (a) Two meshes having a current source in common (b) a supermesh

As shown in Fig. 42 (b), we create a supermesh as the periphery of the two meshes and treat it differently. (If a circuit has two or more supermeshes that intersect, they should be combined to form a larger supermesh.) Why treat the supermesh differently? Because mesh analysis applies KVL—which requires that we know the voltage across each branch—and we do not know the voltage across a current source in advance. However, a supermesh must satisfy KVL like any other mesh. Therefore, applying KVL to the supermesh in Fig. 42 (b) gives

$$-20 + 6i_1 + 10i_2 + 4i_2 = 0$$

$$6i_1 + 14i_2 = 20$$

We apply KCL to a node in the branch where the two meshes intersect. Applying KCL to node 0 in Fig. 42 (a) gives

$i_2 = i_1 + 6$
Then by solving the previous equations
$i_1 = -3.2$ A
$i_2 = 2.8$ A

Note the following properties of a supermesh:
1. The current source in the supermesh provides the constraint equation necessary to solve for the mesh currents.
2. A supermesh has no current of its own.
3. A supermesh requires the application of both KVL and KCL.

EX: Use mesh analysis to determine i_1, i_2 and i_3 in Fig. 43.

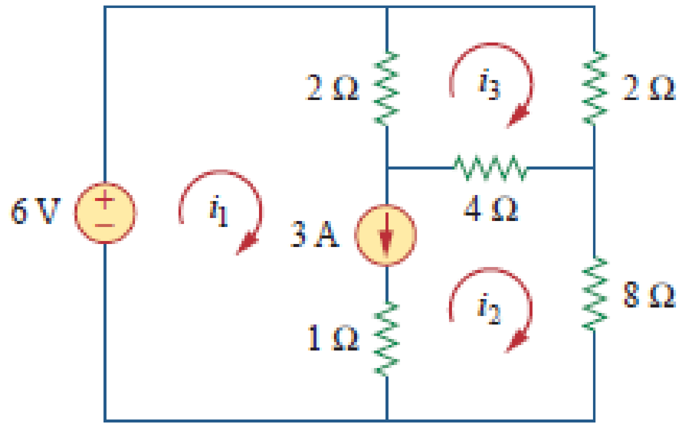


Fig. 43

Answer: i_1 = 3.474 A, i_2 = 0.4737 A, i_3 = 1.1052 A.

Superposition

If a circuit has two or more independent sources, one way to determine the value of a specific variable (voltage or current) is to use nodal or mesh analysis. Another way is to determine the contribution of each independent source to the variable and then add them up. The latter approach is known as the superposition. The idea of superposition rests on the linearity property. The superposition principle states that the voltage across (or current through) an element in a linear circuit is the algebraic sum of the voltages across (or currents through) that element due to each independent source acting alone. The principle of superposition helps us to analyze a linear circuit with more than one independent source by calculating the contribution of each independent source separately. However, to apply the superposition principle, we must keep two things in mind:

1. We consider one independent source at a time while all other independent sources are turned off. This implies that we replace every voltage source by 0 V (or a short circuit), and every current source by 0 A (or an open circuit). This way we obtain a simpler and more manageable circuit.
2. Dependent sources are left intact because they are controlled by circuit variables. With these in mind, we apply the superposition principle in three steps.

Steps to Apply Superposition Principle:

1. Turn off all independent sources except one source. Find the output (voltage or current) due to that active source using the techniques
2. Repeat step 1 for each of the other independent sources.
3. Find the total contribution by adding algebraically all the contributions due to the independent sources.

Analyzing a circuit using superposition has one major disadvantage:

it may very likely involve more work. If the circuit has three independent sources, we may have to analyze three simpler circuits each providing the contribution due to the respective individual source.

However, superposition does help reduce a complex circuit to simpler circuits through replacement of voltage sources by short circuits and of current sources by open circuits. Keep in mind that superposition is based on linearity. For this reason, it is not applicable to the effect on power due to each source, because the power absorbed by a resistor depends on the square of the voltage or current. If the power value is needed, the current through (or voltage across) the element must be calculated first using superposition.

Example: Use the superposition theorem to find v in the circuit of Fig. 44.

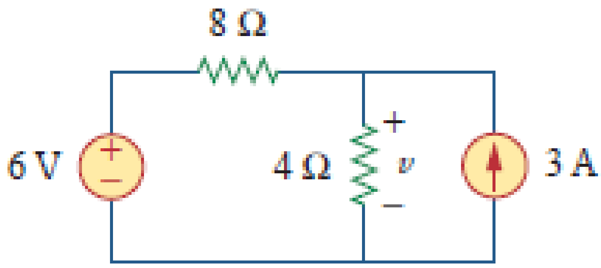


Fig. 44

Solution:

Since there are two sources, let

$V = v_1 + v_2$

Where v_1 and v_2 are the contributions due to the 6-V voltage source and the 3-A current source, respectively. To obtain v_1 we set the current source to zero, as shown in Fig. 45 (a). Applying KVL to the loop in Fig. 45 (a) gives

$$12i_1 - 6 = 0 \quad \Rightarrow \quad i_1 = 0.5\ \text{A}$$

Thus,

$V_1 = 4\, i_1 = 2$ V

To get V_2, we set the voltage source to zero as in Fig. 45 (b) using current division

$$i_3 = \frac{8}{4+8}(3) = 2\ \text{A}$$

Hence,

$V_2 = 4\, i_3 = 8$ V

And we find

$V = V_1 + V_2 = 2 + 8 = 10$ V

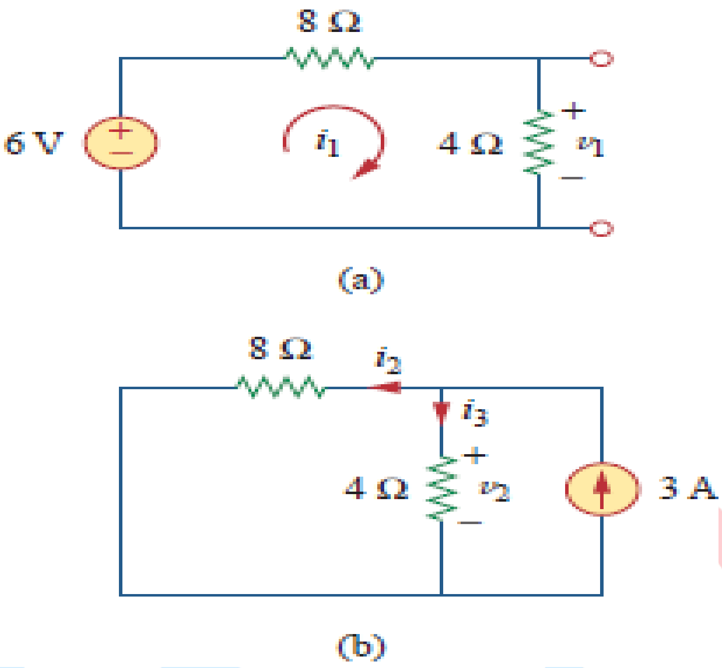


Fig. 45

Example: Using the superposition theorem, find in the circuit of Fig. 46.

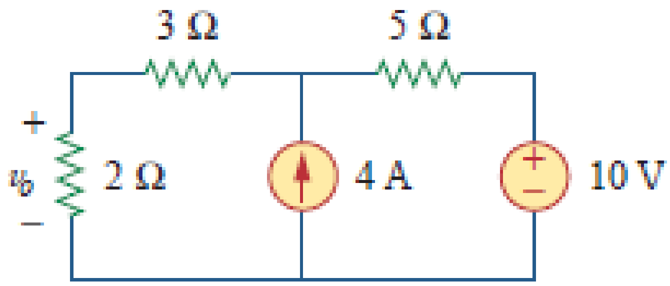


Fig. 46

Answer: 6 V.

Thevenin's Theorem

It often occurs in practice that a particular element in a circuit is variable (usually called the load) while other elements are fixed. As a typical example, a household outlet terminal may be connected to different appliances constituting a variable load. Each time the variable element is changed, the entire circuit has to be analyzed all over again. To avoid this problem, Thevenin's theorem provides a technique by which the fixed part of the circuit is replaced by an equivalent circuit. According to Thevenin's theorem, the linear circuit in Fig. 47 (a) can be replaced by that in Fig. 47 (b). (The load in Fig. 47 may be a single resistor or another circuit.) The circuit to the left of the terminals a-b in Fig.

47 (b) is known as the Thevenin equivalent circuit; it was developed in 1883 by M. Leon Thevenin (1857–1926), a French telegraph engineer.

Thevenin's theorem states that a linear two-terminal circuit can be replaced by an equivalent circuit consisting of a voltage source V_{Th} in series with a resistor R_{Th}, where V_{Th} is the open-circuit voltage at the terminals and R_{Th} is the input or equivalent resistance at the terminals when the independent sources are turned off.

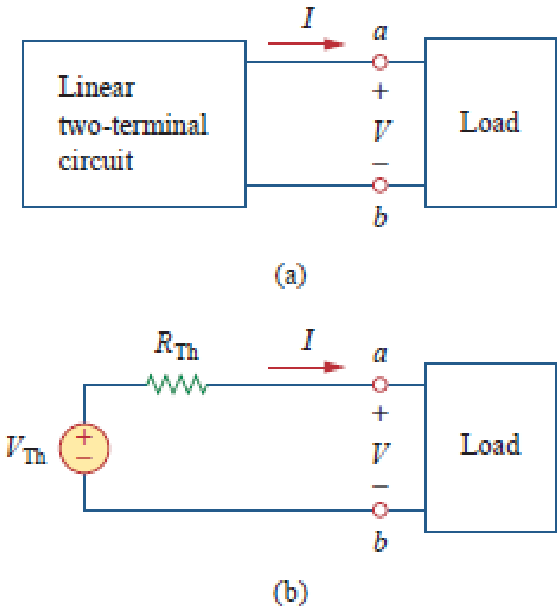


Fig. 47 Replacing a linear two terminal circuit by its Thevenin equivalent (a) original (b) the Thevenin equivalent

Our major concern right now is how to find the Thevenin equivalent voltage V_{th} and resistance R_{th} . To do so, suppose the two circuits in Fig. 47 are equivalent. Two circuits are said to be equivalent if they have the same voltage-current relation at their terminals. Let us find out what will make the two circuits in Fig. 47 equivalent. If the terminals are made open-circuited (by removing the load), no current flows, so that the open-circuit voltage across the terminals in Fig. 47 (a) must be equal to the voltage source in Fig. 47 (b), since the two circuits are equivalent. Thus is the open-circuit voltage across the terminals as shown in Fig. 48 (a); that is,

$V_{th} = V_{oc}$

Again, with the load disconnected and terminals open circuited, we turn off all independent sources. The input resistance (or equivalent resistance) of the dead circuit at the terminals a-b in Fig. 47 (a) must be equal to R_{th} in Fig. 47 (b) because the two circuits are equivalent. Thus, R_{th} is the input resistance at the terminals when the independent sources are turned off, as shown in Fig. 48(b); that is,

$R_{th} = R_{in}$

If the network has no dependent sources, we turn off all independent sources. R_{th} is the input resistance of the network looking between terminals a and b, as shown in Fig. 48 (b).

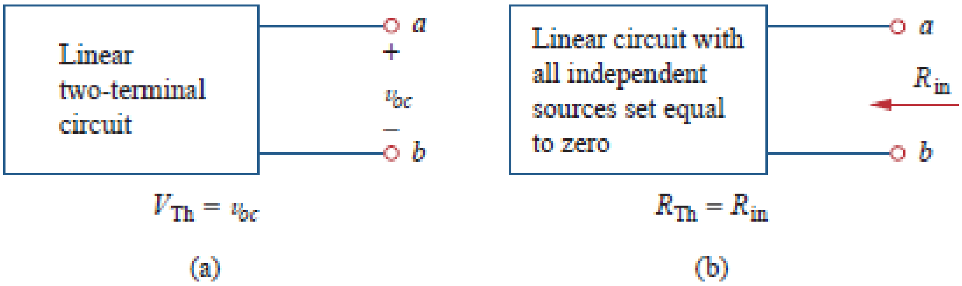


Fig. 48 Finding V_{TH} and R_{TH}

EX

Find the Thevenin equivalent circuit of the circuit shown in Fig. 49, to the left of the terminals a-b Then find the current through R_L= 6 Ω, 16 Ω and 36 Ω

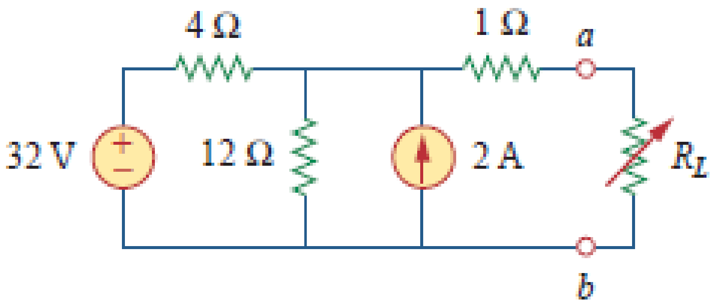


Fig. 49

Solution:

We find R_{th} by turning off the 32-V voltage source (replacing it with a short circuit) and the 2-A current source (replacing it with an open circuit). The circuit becomes what is shown in Fig. 50 (a).

Thus,

$$R_{Th} = 4 \parallel 12 + 1 = \frac{4 \times 12}{16} + 1 = 4\ \Omega$$

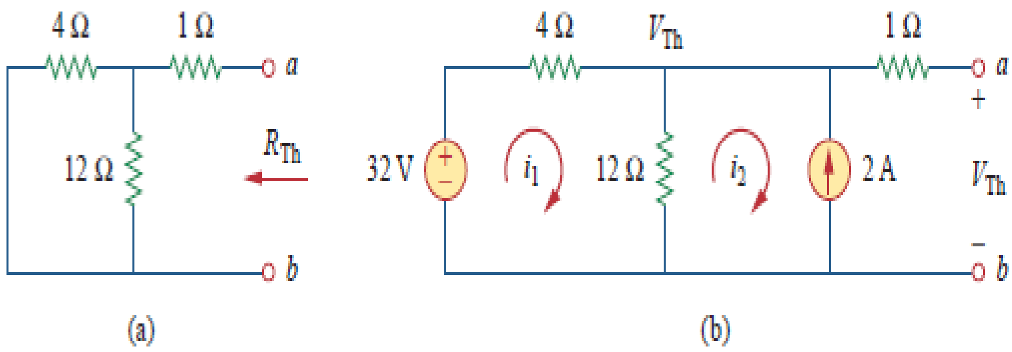


Fig. 50

To find V_{TH}, Applying mesh analysis to the two loops in Fig. 50 (b) we obtain

$-32 + 4i_1 + 12 (i_1 - i_2) = 0$

$i_2 = -2$ A

Then

$i_1 = 0.5$ A

$V_{TH} = 12 (i_1 - i_2) = 30$ V

$I_L = 30 / (R_L + R_{TH}) = 30 / (R_L + 4)$

When $R_L = 6$,

$I_L = 30 / 10 = 3$ A

When $R_L = 16$

$I_L = 30/20 = 1.5$ A

Finally at $R_L = 36$,

$I_L = 0.75$ A

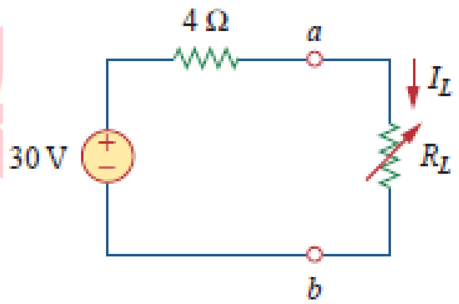


Fig. 60

Note that in Fig. 61 the Maximum power is transferred to the load when the load resistance equals the Thevenin resistance as seen from the load ($R_L = R_{Th}$). It can be proved by getting P_{load} as a function of R_L from Fig. 62. Then differentiate the P_{load} equation with respect to R_L. Finally, put the differentiation output equal to zero.

$$p = i^2 R_L = \left(\frac{V_{Th}}{R_{Th} + R_L} \right)^2 R_L$$

$$\boxed{p_{max} = \frac{V_{Th}^2}{4R_{Th}}}$$

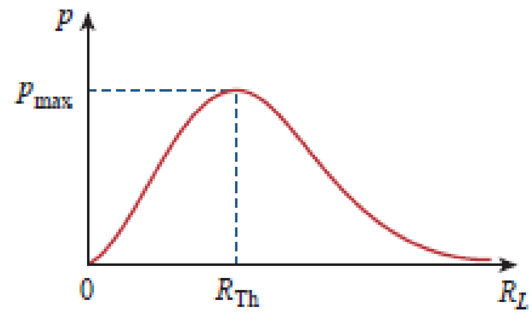


Fig. 61 Power delivered to the load as function of R_L

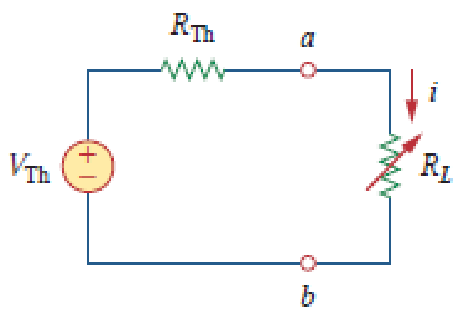


Fig. 62 Circuit used for Maximum power transfer

EX : Using Thevenin's theorem, find the equivalent circuit to the left of the terminals in the circuit of Fig. 63. Then find I.

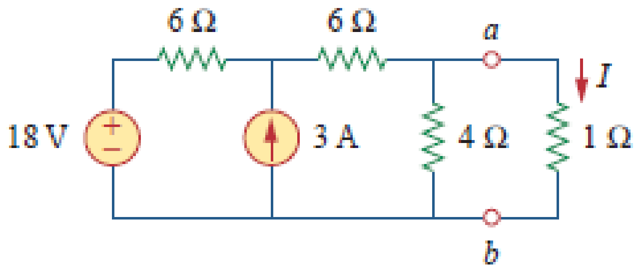


Fig. 63

Answer: $V_{Th} = 9$ V, $R_{Th} = 3\ \Omega$, $I = 2.25$ A.

Chapter 3

Single-phase AC electric circuits

Objective

- To understand the basic principles of Ac electric circuit.
- To study different techniques to analysis of AC electric circuits.

Overview

Thus far our analysis has been limited for the most part to dc circuits: those circuits excited by constant or time-invariant sources. We have restricted the forcing function to dc sources for the sake of simplicity, for pedagogic reasons, and also for historic reasons. Historically, dc sources were the main means of providing electric power up until the late 1800s. At the end of that century, the battle of direct current versus alternating current began. Both had their advocates among the electrical engineers of the time. Because ac is more efficient and economical to transmit over long distances, ac systems ended up the winner. Thus, it is in keeping with the historical sequence of events that we considered dc sources first. We now begin the analysis of circuits in which the source voltage or current is time-varying. In this chapter, we are particularly interested in sinusoidally time-varying excitation, or simply, excitation by a sinusoid. A sinusoid is a signal that has the form of the sine or cosine function. We begin with a basic discussion of sinusoids and phasors. We then introduce the concepts of impedance and admittance. The basic circuit laws, Kirchhoff's and Ohm's, introduced for dc circuits, will be applied to ac circuits. Consider the sinusoidal voltage

$V(t) = V_m \sin \omega t$ (1)

Where the sinusoid is shown in Fig. 1(a) as a function of its argument ω and in Fig. 1 as a function of time. It is evident that the sinusoid repeats itself every T seconds; thus, T is called the period of the sinusoid. From the two plots in Fig. 1, we observe that

$T = 2\pi / \omega$ (2)

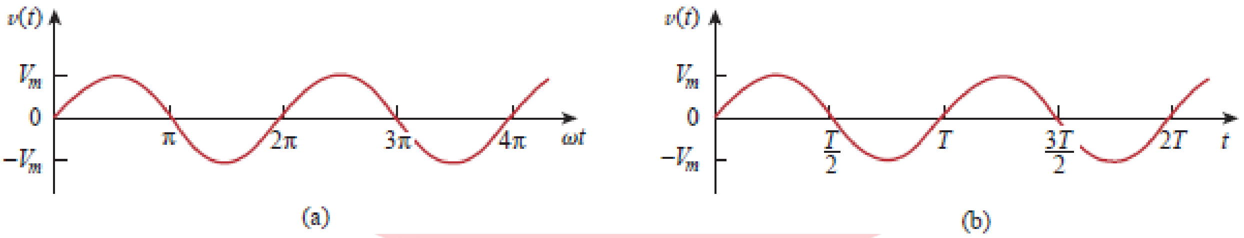


Fig. 1 A sketch of $V_m \sin \omega t$ as a function of (a) ωt (b) t

The fact that V(t) repeats itself every T seconds is shown by replacing t by t+T in Eq. (1). We get

$$v(t+T) = V_m \sin\omega(t+T) = V_m \sin\omega\left(t + \frac{2\pi}{\omega}\right)$$
$$= V_m \sin(\omega t + 2\pi) = V_m \sin\omega t = v(t)$$

Hence,

$$\boxed{v(t+T) = v(t)}$$

That is V, has the same value at t+T as it does at t and v(t) is said to be periodic. As mentioned, the period T of the periodic function is the time of one complete cycle or the number of seconds per cycle. The reciprocal of this quantity is the number of cycles per second, known as the cyclic frequency f of the sinusoid. Thus,

$f = 1 / T$ (3)

From Eqs. (2) and (3), it is clear that

$\omega = 2\pi f$ (4)

While ω is in radians per second (rad/s), f is in hertz (Hz). Let us now consider a more general expression for the sinusoid,

$V(t) = V_m \sin(\omega t + \varphi)$ (5)

Where $(\omega t + \varphi)$ is the argument and φ is the phase. Both argument and phase can be in radians or degrees. Let us examine the two sinusoids shown in Fig. 2.

$V_1(t) = V_m \sin \omega t$ and $V_2(t) = V_m \sin(\omega t + \varphi)$ (6)

The starting point of in Fig. 2 occurs first in time. Therefore, we say that V_2 leads V_1 by φ or that V1 lags V2 by φ If we also say that V_1 and V_2 are out of phase. If then $\varphi = 0$, V_1 and V_2 are said to be in phase; they reach their minimum and maximum at exactly the same time. We can compare V_1 and V_2 in this manner because they operate at the same frequency; they do not need to have the same amplitude.

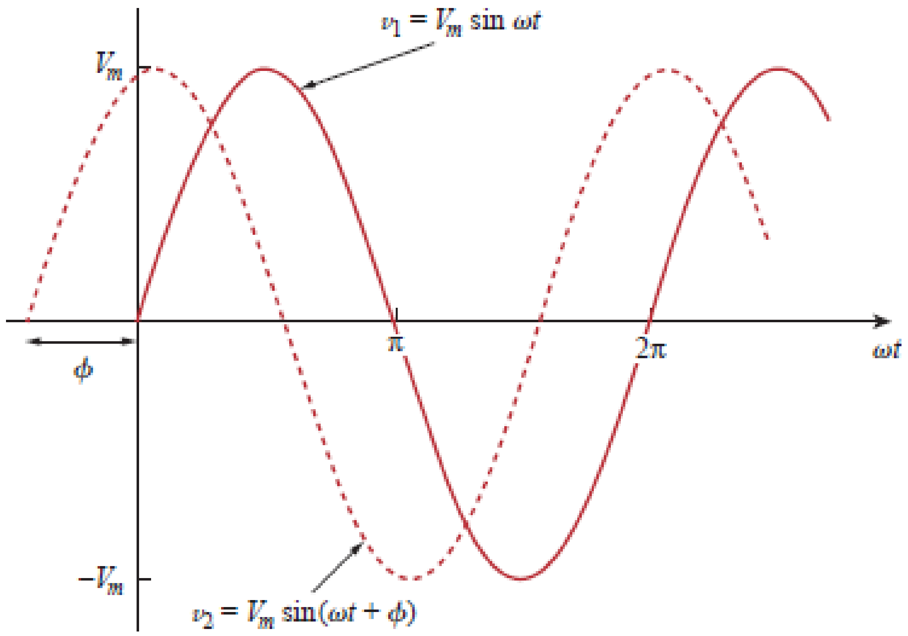


Fig. 2 Two sinusoids with different phases

A sinusoid can be expressed in either sine or cosine form. When comparing two sinusoids, it is expedient to express both as either sine or cosine with positive amplitudes. Using the following relationships, we can transform a sinusoid from sine form to cosine form or vice versa.

$\sin(\omega t \pm 180^\circ) = -\sin \omega t$

$\cos(\omega t \pm 180^\circ) = -\cos \omega t$

$\sin(\omega t \pm 90^\circ) = \pm \cos \omega t$

$\cos(\omega t \pm 90^\circ) = \mp \sin \omega t$

Ex : Find the amplitude, phase, period, and frequency of the sinusoid

$$v(t) = 12\cos(50t + 10^\circ)$$

Solution:

The amplitude is $V_m = 12$ V.

The phase is $\phi = 10^\circ$.

The angular frequency is $\omega = 50$ rad/s.

The period $T = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{50} = 0.1257$ s.

The frequency is $f = \dfrac{1}{T} = 7.958$ Hz.

Example: Calculate the phase angle between $V_1 = -10\cos(\omega t + 50^\circ)$ and $V_2 = 12\sin(\omega t - 10^\circ)$

Solution

$$v_1 = -10\cos(\omega t + 50^\circ) = 10\cos(\omega t + 50^\circ - 180^\circ)$$

$$v_1 = 10\cos(\omega t - 130^\circ) \quad \text{or} \quad v_1 = 10\cos(\omega t + 230^\circ)$$

and

$$v_2 = 12\sin(\omega t - 10^\circ) = 12\cos(\omega t - 10^\circ - 90^\circ)$$

$$v_2 = 12\cos(\omega t - 100^\circ)$$

Thus

The phase difference between V_1 and V_2 is 30°

Phasors

Sinusoids are easily expressed in terms of phasors, which are more convenient to work with than sine and cosine functions. A phasor is a complex number that represents the amplitude and phase of a sinusoid. Phasors provide a simple means of analyzing linear circuits excited by sinusoidal sources; solutions of such circuits would be intractable otherwise. The notion of solving Ac circuits using phasors was first introduced by Charles Steinmetz in 1893. Before we completely define phasors and apply them to circuit analysis, we need to be thoroughly familiar with complex numbers.

A complex number z can be written in rectangular form as

$Z = x + j\,y$ (7)

Where $j=\sqrt{-1}$; x is the real part of z; y is the imaginary part of z. In this context, the variables x and y do not represent a location as in two-dimensional vector analysis but rather the real and imaginary parts of z in the complex plane. Nevertheless, we note that there are some resemblances between manipulating complex numbers and manipulating two-dimensional

vectors. The complex number z can also be written in polar or exponential form as
$z = r\angle\varphi = re^{j\varphi}$
Where r is the magnitude of z, and φ is the phase of z. We notice that z can be represented in three ways:

$$z = x + jy = r\angle\phi = r(\cos\phi + j\sin\phi)$$

The Idea of phasors representation is based on Euler's identity. In general

$e^{-j\varphi} = \cos\varphi \pm j\sin\varphi$ (8)

Then

$\cos\varphi = \mathrm{Re}\,(e^{-j\varphi})$ (9a)

$\sin\varphi = \mathrm{Im}\,(e^{-j\varphi})$ (9b)

Thus V(t) can be presented as

$V(t) = V_m\cos(\omega t + \varphi) = \mathrm{Re}\,(V_m e^{j(\omega t+\varphi)})$ (10)

Or

$V(t) = \mathrm{Re}\,(V_m e^{j\varphi} e^{j\omega t})$ (11)

Thus

$V(t) = \mathrm{Re}\,(V e^{j\omega t})$ (12)

Where

$V = V_m e^{j\varphi} = V_m\angle\varphi$ (13)

V is thus the phasor representation of the sinusoid V(t). In other words, a phasor is a complex representation of the magnitude and phase of a sinusoid. Either Eq. (9a) or Eq. (9b) can be used to develop the phasor, but the standard convention is to use Eq. (9a). One way of looking at Eqs. (12) and (13) is to consider the plot of the sinor $Ve^{j\omega t} = V_m e^{j(\omega t+\varphi)}$ on the complex plane. As time increases, the sinor rotates on a circle of radius V_m at an angular velocity ω in the counterclockwise direction, as shown in Fig. 3(a). We may regard v(t) as the projection of the sinor $Ve^{j\omega t}$ on the real axis, as shown in Fig. 3(b). The value of the sinor at time t=0 is the phasor V of the sinusoid V(t). The sinor may be regarded as a rotating phasor. Thus, whenever a sinusoid is expressed as a phasor, the term $e^{j\omega t}$ is implicitly present. It is therefore important, when dealing with phasors, to keep in mind the frequency ω of the phasor; otherwise we can make serious mistakes.

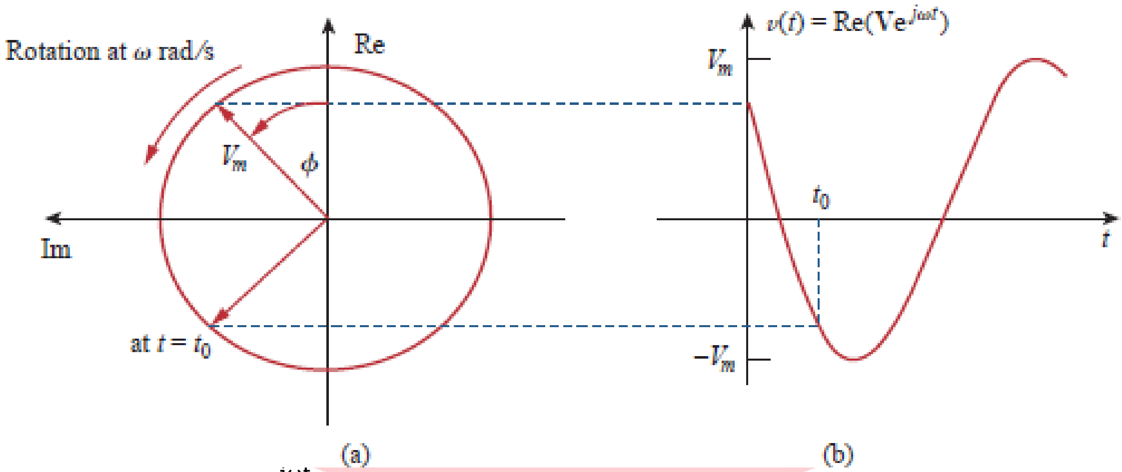


Fig. 3 Representation of $Ve^{j\omega t}$ (a)sinor rotating counterclockwise (b) its projection on the real axis

Equation (12) states that to obtain the sinusoid corresponding to a given phasor V, multiply the phasor by the time factor $e^{j\omega t}$ and take the real part. As a complex quantity, a phasor may be expressed in rectangular form, polar form, or exponential form. Since a phasor has magnitude and phase ("direction"), it behaves as a vector and is printed in boldface. Equations (10) through (12) reveal that to get the phasor corresponding to a sinusoid, we first express the sinusoid in the cosine form so that the sinusoid can be written as the real part of a complex number. Then we take out the time factor $e^{j\omega t}$ and whatever is left is the phasor corresponding to the sinusoid. By suppressing the time factor, we transform the sinusoid from the time domain to the phasor domain. This transformation is summarized as follows:

$$v(t) = V_m \cos(\omega t + \phi) \quad \Leftrightarrow \quad \mathbf{V} = V_m \angle \phi$$

(Time-domain representation) (Phasor-domain representation)

Table 1 Sinusoid phasor transformation

Time domain representation	Phasor domain representation
$V_m \cos(\omega t + \phi)$	$V_m \angle \phi$
$V_m \sin(\omega t + \phi)$	$V_m \angle \phi - 90^\circ$
$I_m \cos(\omega t + \theta)$	$I_m \angle \theta$
$I_m \sin(\omega t + \theta)$	$I_m \angle \theta - 90^\circ$

This shows that the derivative of V(t) is transformed to the phasor domain as jωV

$$\frac{dv}{dt} \quad \Leftrightarrow \quad j\omega \mathbf{V}$$

(Time domain) (Phasor domain)

Similarly, the integral of V(t) is transformed to the phasor domain as V/jω

$$\int v\, dt \quad \Leftrightarrow \quad \frac{\mathbf{V}}{j\omega}$$

(Time domain) (Phasor domain)

Example Transform these sinusoids to phasors

(a) $i = 6 \cos(50t - 40^\circ)$ A

(b) $v = -4 \sin(30t + 50^\circ)$ V

Solution

(a) $i = 6 \cos(50t - 40^\circ)$ has the phasor

$$\mathbf{I} = 6\angle -40^\circ \text{ A}$$

(b) Since $-\sin A = \cos(A + 90^\circ)$,

$$v = -4 \sin(30t + 50^\circ) = 4 \cos(30t + 50^\circ + 90^\circ)$$
$$= 4 \cos(30t + 140^\circ) \text{ V}$$

The phasor form of v is

$$\mathbf{V} = 4\angle 140^\circ \text{ V}$$

Phasor Relationships for Circuit Elements

Now that we know how to represent a voltage or current in the phasor or frequency domain, one may legitimately ask how we apply this to circuits involving the passive elements R, L, and C. What we need to do is to transform the voltage-current relationship from the time domain to the frequency domain for each element. Again, we will assume the passive sign convention. We begin with the resistor. If the current through a resistor R is i= Im cos(ωt+ φ) the voltage across it is given by Ohm's law as

V =I R = R Im cos(ωt+ φ)

The phasor form of voltage and current

V =R Im $\angle\varphi$ (14)

I = Im $\angle\varphi$ (15)

showing that the voltage-current relation for the resistor in the phasor domain continues to be Ohm's law, as in the time domain. Figure 4 illustrates the voltage-current relations of a resistor. We should note from Eqs. (14) and (15) that voltage and current are in phase, as illustrated in the phasor diagram in Fig. 5.

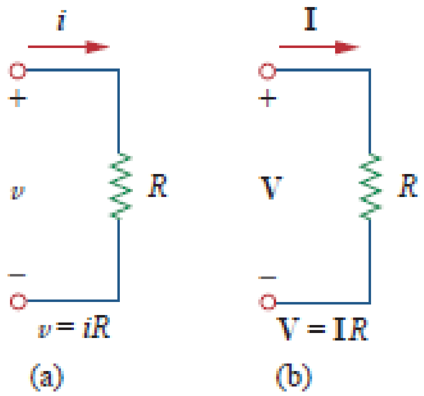


Fig. 4 Voltage current relation for a resistor in (a) time domain (b) frequency domain

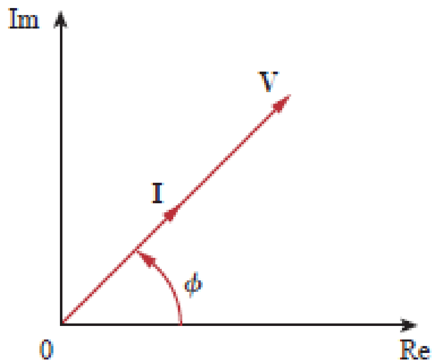


Fig. 5 Phasor diagram for the resistor

For the inductor L, assume the current through it is i= Im cos(ωt+ φ) The voltage across the inductor is

V = L di/dt $= -\omega L Im \sin(\omega t + \varphi)$

$= \omega L Im \cos(\omega t + \varphi + 90°)$

Then

V= $\omega L Im \angle\varphi + 90$

But I = Im $\angle\varphi$

Thus

V = j ω L I where $e^{j90°}$ = j

Figure 6 illustrates the voltage-current relations of an inductor.

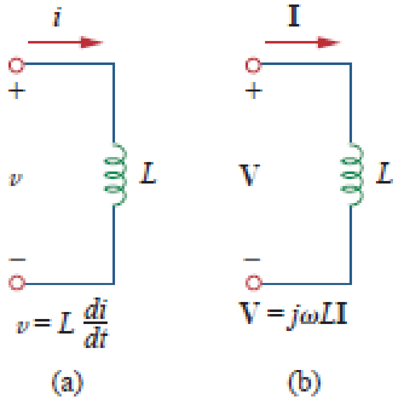


Fig. 6 voltage current relations for an inductor in the (a) time domain (b) frequency domain

Showing that the voltage has a magnitude ω L I_m of and a phase of φ + 90˚ The voltage and current are 90˚ out of phase. Specifically, the current lags the voltage by 90˚.
For the capacitor C, assume the voltage across it is V = V_m cos (ω t + φ). The current through the capacitor is
I= C dv/dt
Similarly we can get that,
I = j ω C V
Then
V = I / jωc
Showing that the current and voltage are 90˚ out of phase. To be specific, the current leads the voltage by 90˚. Figure 7 illustrates the voltage-current relations of a capacitor.

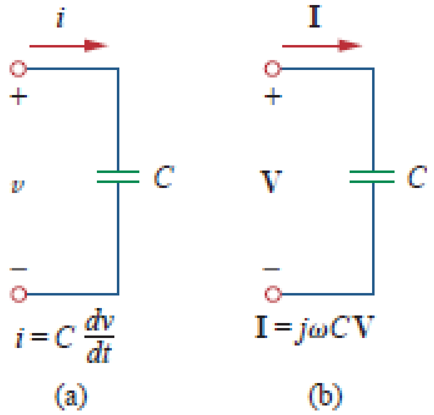


Fig. 7 voltage current relations for a capacitor in the (a) time domain (b) frequency domain

Table 2 summarized the voltage current relationships.

Table 2 summary

Element	Time domain	Frequency domain
R	$v = Ri$	$\mathbf{V} = R\mathbf{I}$
L	$v = L\frac{di}{dt}$	$\mathbf{V} = j\omega L\mathbf{I}$
C	$i = C\frac{dv}{dt}$	$\mathbf{V} = \frac{\mathbf{I}}{j\omega C}$

Example: The voltage V = 12 cos (60t +45°) is applied to a 0.1 H inductor Find the current through the inductor.

Solution

$V = j\omega LI$

$\omega = 60$ rad /s

$V = 12 \angle 45$

$I = V / j\omega l = 12 \angle 45 / (j60*0.1) = 2 \angle -45$ A

$i_l(t) = 2 \cos(60t - 45°)$ A

Example: The voltage V = 10 cos (100t +30°) is applied to a 50μF capacitor. Find the current through the capacitor.

Answer: 50 cos (100t +120°) mA

Impedance and Admittance

The impedance Z of a circuit is the ratio of the phasor voltage V to the phasor current I, measured in ohms (Ω). In the preceding section, we obtained the voltage-current relations for the three passive elements as

$$\mathbf{V} = R\mathbf{I}, \qquad \mathbf{V} = j\omega L\mathbf{I}, \qquad \mathbf{V} = \frac{\mathbf{I}}{j\omega C}$$

Or

$$\frac{\mathbf{V}}{\mathbf{I}} = R, \qquad \frac{\mathbf{V}}{\mathbf{I}} = j\omega L, \qquad \frac{\mathbf{V}}{\mathbf{I}} = \frac{1}{j\omega C}$$

Then Ohm's law in phasor form for any type of element

$$\boxed{\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} \quad \text{or} \quad \mathbf{V} = \mathbf{Z}\mathbf{I}}$$

Where Z is a frequency-dependent quantity known as impedance, measured in ohms.
From the previous equations the impedance of resistor, inductor and capacitor are $Z_R = R$, $Z_L = j\omega L$ and $Z_c = -J/\omega C$ respectively. At dc ω=o then ZL = 0 and $Z_c \rightarrow \infty$

Generally

$Z = R + j X$

The impedance may also be expressed in polar form as

$Z = Z \angle \theta$

The admittance Y is the reciprocal of impedance, measured in siemens (S).

$$\boxed{\mathbf{Y} = G + jB}$$

Where G is called the conductance and B is called the susceptance. Admittance, conductance, and susceptance are all expressed in the unit of siemens (or mhos). Table 3 shows the summary.

Table 3 Impedance and admittance of passive elements

Element	Impedance	Admittance
R	$\mathbf{Z} = R$	$\mathbf{Y} = \frac{1}{R}$
L	$\mathbf{Z} = j\omega L$	$\mathbf{Y} = \frac{1}{j\omega L}$
C	$\mathbf{Z} = \frac{1}{j\omega C}$	$\mathbf{Y} = j\omega C$

Example: Find V(t) and i (t) in the circuit shown in Fig. 8
Solution
V = 10 cos 4t, ω = 4
Vs = 10 $\angle 0^\circ$ V
Z= 5 + [1 /jωc] = 5 + 1 / (j4*0.1) = 5 - j 2.5 Ω
I= Vs / Z = 1.6 + j 0.8 = 1.789 $\angle 26.57^\circ$ A
Voltage across the capacitor
V = I Zc = I / jωc = 4.47 $\angle -63.43^\circ$ V
I(t) = 1.789 cos(4t + 26.57°) A
V(t) = 4.47 cos(4t - 63.43°) V

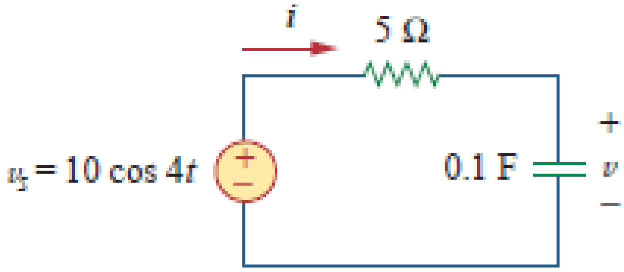


Fig. 8

Example: Refer to Fig. 9 Determine V (t),i(t)

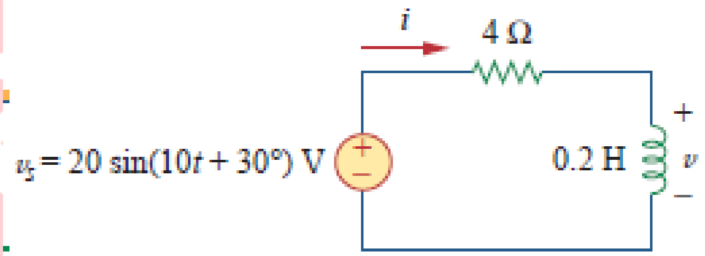


Fig. 9

Answer: 8.944 sin(10t + 93.43°) V , 4.472 sin(10t + 3.43°) A

Kirchhoff's Laws in the Frequency Domain

We cannot do circuit analysis in the frequency domain without Kirchhoff's current and voltage laws. The Kirchhoff's voltage and current law hold for phasors. For KVL, let V_1, V_2,……V_n are the phasors form of voltages around a closed loop.
Then
$V_1 + V_2 + \ldots\ldots\ldots V_n = 0$

If we let I_1, I_2, ...,I_n are the phasors form of current leaving or entering a closed surface then

$I_1+I_2+............I_N=0$

Consider the N series-connected impedances shown in Fig. 10. The same current "I" flows through the impedances.

$$\boxed{\mathbf{Z}_{eq} = \mathbf{Z}_1 + \mathbf{Z}_2 + \cdots + \mathbf{Z}_N}$$

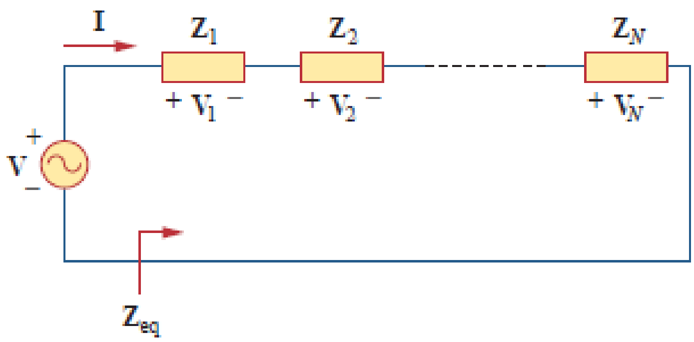


Fig.10 " N" impedance in series

Showing that the total or equivalent impedance of series-connected impedances is the sum of the individual impedances. This is similar to the series connection of resistances. If N =2 as shown in Fig. 11, the current through the impedances Is

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}_1 + \mathbf{Z}_2}$$

Since $\mathbf{V}_1 = \mathbf{Z}_1\mathbf{I}$ and $\mathbf{V}_2 = \mathbf{Z}_2\mathbf{I}$, then

$$\boxed{\mathbf{V}_1 = \frac{\mathbf{Z}_1}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{V}, \qquad \mathbf{V}_2 = \frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{V}}$$

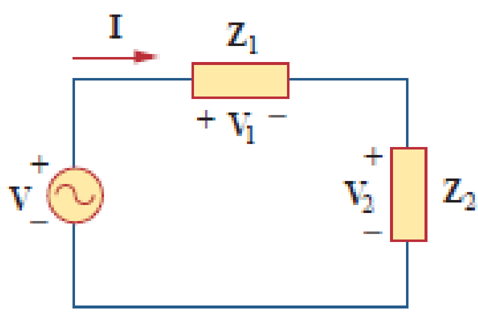


Fig. 11 Voltage division

Which is the voltage-division relationship. In the same manner, we can obtain the equivalent impedance or admittance of the N parallel-connected impedances shown in Fig. 12. The voltage across each impedance is the same. Applying KCL at the top node,

$$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 + \cdots + \mathbf{I}_N = \mathbf{V}\left(\frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \cdots + \frac{1}{\mathbf{Z}_N}\right)$$

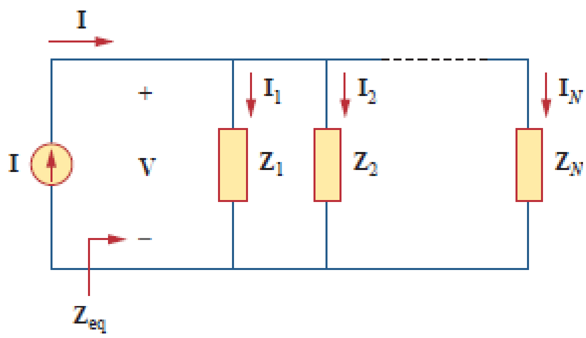


Fig.12 ” N” impedance in series

The equivalent impedance is

$$\frac{1}{\mathbf{Z}_{eq}} = \frac{\mathbf{I}}{\mathbf{V}} = \frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \cdots + \frac{1}{\mathbf{Z}_N}$$

and the equivalent admittance is

$$\boxed{\mathbf{Y}_{eq} = \mathbf{Y}_1 + \mathbf{Y}_2 + \cdots + \mathbf{Y}_N}$$

This indicates that the equivalent admittance of a parallel connection of admittances is the sum of the individual admittances. When as N=2 shown in Fig. 13, the currents in the impedances are

$$\boxed{\mathbf{I}_1 = \frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{I}, \qquad \mathbf{I}_2 = \frac{\mathbf{Z}_1}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{I}}$$

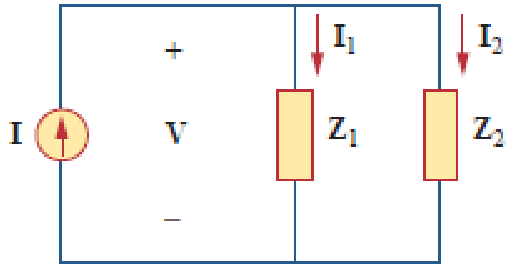


Fig. 13 Current division

Example: Find the input impedance in Fig. 14. Assume that the circuit operates at ω = 50 rad /s.

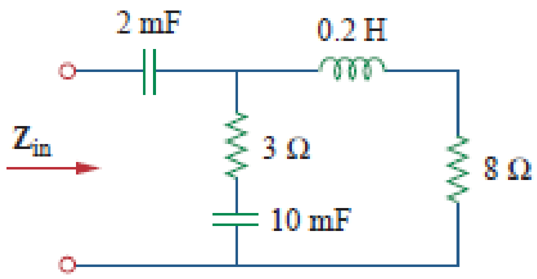


Fig. 14

Solution

Let

$\mathbf{Z}_1$ = Impedance of the 2-mF capacitor

$\mathbf{Z}_2$ = Impedance of the 3-Ω resistor in series with the10-mF capacitor

$\mathbf{Z}_3$ = Impedance of the 0.2-H inductor in series with the 8-Ω resistor

Then

$$\mathbf{Z}_1 = \frac{1}{j\omega C} = \frac{1}{j50 \times 2 \times 10^{-3}} = -j10\ \Omega$$

$$\mathbf{Z}_2 = 3 + \frac{1}{j\omega C} = 3 + \frac{1}{j50 \times 10 \times 10^{-3}} = (3 - j2)\ \Omega$$

$$\mathbf{Z}_3 = 8 + j\omega L = 8 + j50 \times 0.2 = (8 + j10)\ \Omega$$

The input impedance is

$$\mathbf{Z}_{in} = \mathbf{Z}_1 + \mathbf{Z}_2 \parallel \mathbf{Z}_3 = -j10 + \frac{(3 - j2)(8 + j10)}{11 + j8}$$

$$= -j10 + \frac{(44 + j14)(11 - j8)}{11^2 + 8^2} = -j10 + 3.22 - j1.07\ \Omega$$

Thus,

$$\mathbf{Z}_{in} = 3.22 - j11.07\ \Omega$$

Example: Determine the input impedance of the circuit in Fig. 15 at ω = 10 rad/s.

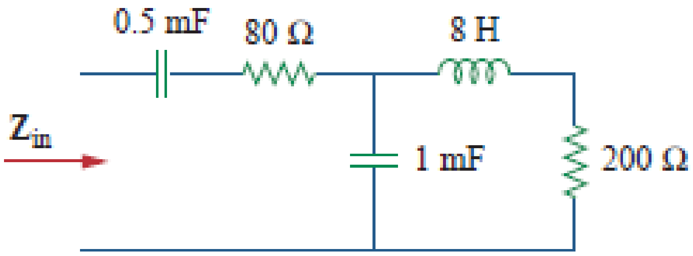


Fig. 15

Answer; 129.52 – j 295 Ω

Example: calculate V_o in the circuit of Figure 16

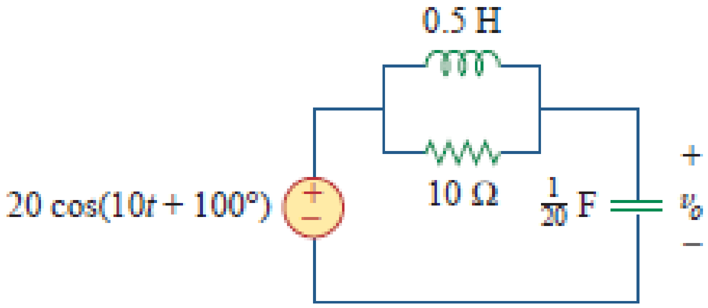


Fig. 16

Answer: $v_o(t) = 14.142 \cos(10t - 35^\circ)$ V

Analyze AC Circuits

Analyzing Ac circuits usually requires three steps. Steps to Analyze AC Circuits:
1. Transform the circuit to the phasor or frequency domain.
2. Solve the problem using circuit techniques (nodal analysis, mesh analysis, superposition, etc.).
3. Transform the resulting phasor to the time domain.

Note, Step 1 is not necessary if the problem is specified in the frequency domain. In step 2, the analysis is performed in the same manner as dc circuit analysis except that complex numbers are involved.

Nodal Analysis

The basis of nodal analysis is Kirchhoff's current law. Since KCL is valid for phasors, we can analyze ac circuits by nodal analysis. The following examples illustrate this.
Example: compute V_1 and V_2 in the circuit of Figure 17

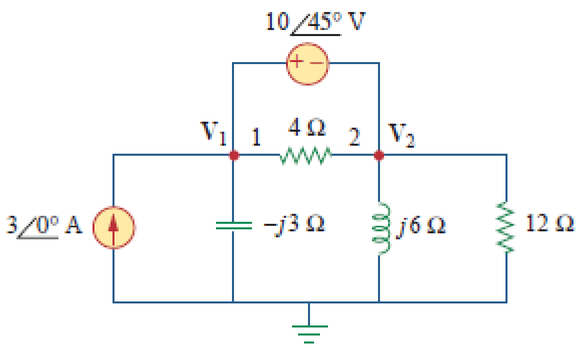


Fig. 17

Solution
Applying KCL at supernode shown in Figure 18 (Nodes 1 and 2)

$$3 = \frac{V_1}{-j3} + \frac{V_2}{j6} + \frac{V_2}{12}$$

$$36 = j4V_1 + (1 - j2)V_2$$

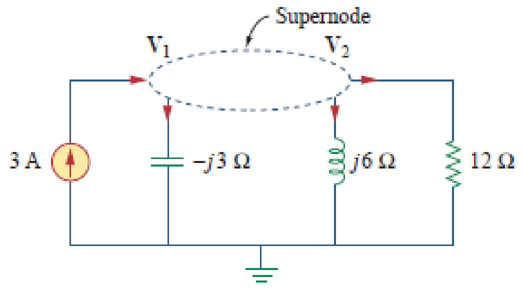


Fig. 18

But a voltage source is connected between nodes 1 and 2 so that

$$V_1 = V_2 + 10\angle 45^\circ$$

Then

$$36 - 40\angle 135^\circ = (1 + j2)V_2 \quad \Rightarrow \quad V_2 = 31.41\angle -87.18^\circ \text{ V}$$

Thus

$$V_1 = V_2 + 10\angle 45^\circ = 25.78\angle -70.48^\circ \text{ V}$$

Example: calculate V_1 and V_2 in the circuit shown in Figure 19

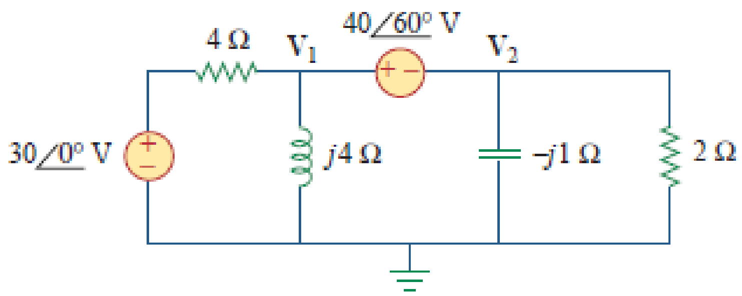


Fig. 19

Answer: $V_1 = 38.72\angle 69.67^\circ \text{ V}, V_2 = 6.752\angle 165.7^\circ \text{ V}.$

Mesh Analysis

Kirchhoff's voltage law (KVL) forms the basis of mesh analysis. The validity of KVL for ac circuits was shown before and is illustrated in the following examples. Keep in mind that the very nature of using mesh analysis is that it is to be applied to planar circuits.

Example Determine current in the circuit of Fig. 20 using mesh analysis.

Solution:

Applying KVL to mesh 1, we obtain

$(8 + j10 - j2)\ \mathbf{I}_1 - (-j2)\ \mathbf{I}_2 - j10\mathbf{I}_3 = 0$

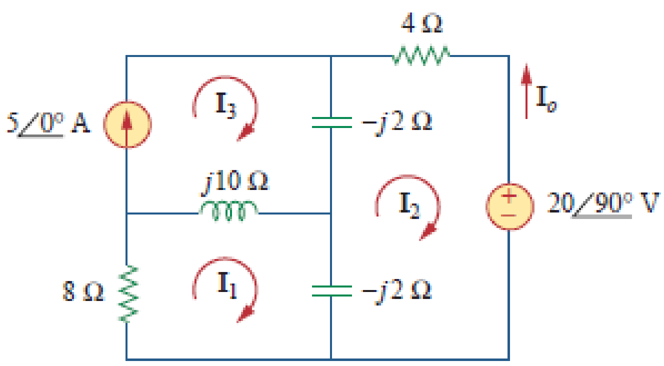


Fig. 20

For mesh 2,

$$(4 - j2 - j2)\mathbf{I}_2 - (-j2)\mathbf{I}_1 - (-j2)\mathbf{I}_3 + 20\angle 90^\circ = 0$$

For mesh 3, $\mathbf{I}_3 = 5$ A

By sub by $\mathbf{I}_3$ in the previous equations

$$(8 + j8)\mathbf{I}_1 + j2\mathbf{I}_2 = j50$$

$$j2\mathbf{I}_1 + (4 - j4)\mathbf{I}_2 = -j20 - j10$$

Put the equations in matrix form

$$\begin{bmatrix} 8 + j8 & j2 \\ j2 & 4 - j4 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} j50 \\ -j30 \end{bmatrix}$$

From which we obtain

$$\Delta = \begin{vmatrix} 8 + j8 & j2 \\ j2 & 4 - j4 \end{vmatrix} = 32(1 + j)(1 - j) + 4 = 68$$

$$\Delta_2 = \begin{vmatrix} 8 + j8 & j50 \\ j2 & -j30 \end{vmatrix} = 340 - j240 = 416.17\angle -35.22^\circ$$

$$\mathbf{I}_2 = \frac{\Delta_2}{\Delta} = \frac{416.17\angle -35.22^\circ}{68} = 6.12\angle -35.22^\circ \text{ A}$$

The desired current is

$$\mathbf{I}_o = -\mathbf{I}_2 = 6.12\angle 144.78^\circ \text{ A}$$

Example: Find Io in Figure 21 using mesh analysis

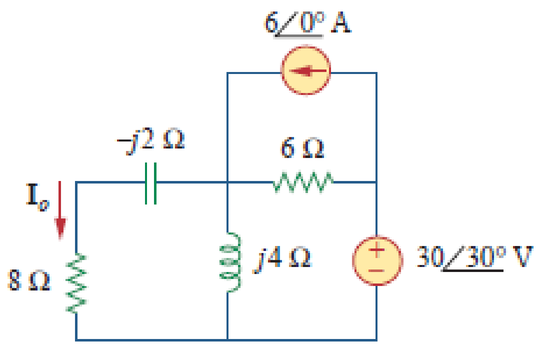


Fig. 21

Answer: $3.582 \angle 65.45^\circ$ A

Example: Find Vo in Figure 22 using mesh analysis

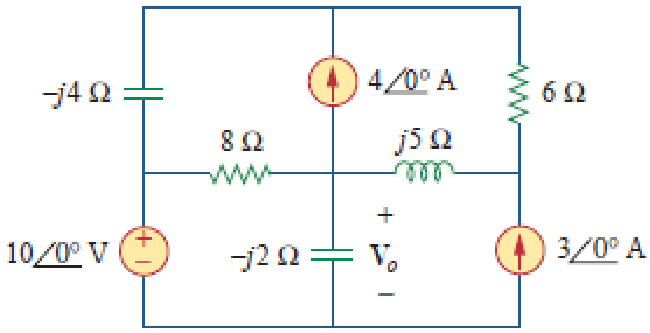


Fig. 22

Solution
As shown in Figure 24, meshes 3 and 4 form a supermesh
For mesh 1, KVL gives

$$-10 + (8 - j2)I_1 - (-j2)\, I_2 - 8\, I_3 = 0$$

For mesh 2,
$I2 = -3$
For the supermesh

$$(6 + j5)I_4 + (8 - j4)I_3 + (-j5)\, I_2 - 8\, I_1 = 0$$

From current source between meshes 3 and 4
$I_4 = I_3 + 4$

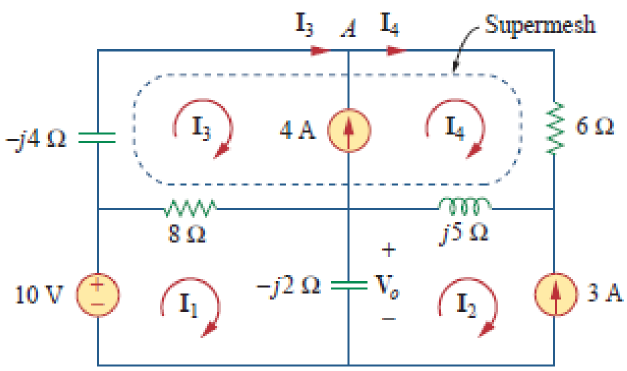


Fig. 24

By solving the previous equations we get
$I_1 = 3.618 \angle 274.5^\circ$ A
$V_o = -j2\,(I_1 - I_2) = -7.2134 - j\,6.568$ V

Superposition Theorem

Since ac circuits are linear, the superposition theorem applies to Ac circuits the same way it applies to dc circuits. The theorem becomes important if the circuit has sources operating at different frequencies. In this case, since the impedances depend on frequency, we must have a different frequency domain circuit for each frequency. The total response must be obtained by adding the individual responses in the time domain. It is incorrect to try to add the responses in the phasor or frequency domain. Why? Because the exponential factor $e^{j\omega t}$ is implicit in sinusoidal analysis, and that factor would change for every angular frequency ω. It would therefore not make sense to add responses at different frequencies in the phasor domain. Thus, when a circuit has sources operating at different frequencies, one must add the responses due to the individual frequencies in the time domain.

Example: Find Vo in Figure 24 using the superposition theorem.

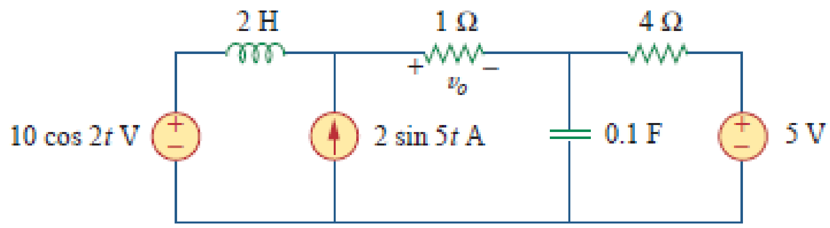


Fig. 24

Solution:

Since the circuit operates at three different frequencies ω=0 for the dc voltage source), one way to obtain a solution is to use superposition, which breaks the problem into single-frequency problems. So we let

$Vo = V_1 + V_2 + V_3$

where V1 is due to the 5-V dc voltage source, V2 is due to the 10 cos2t voltage source, and V3 is due to 2 sin 5t A current source. To find V1, we set to zero all sources except the 5-V dc source. We recall that at steady state, a capacitor is an open circuit to dc while an inductor is a short circuit to dc. There is an alternative way of looking at this. Since ω =0, jωL =0, 1/jωc = ∞ either way, the equivalent circuit is as shown in Fig. 25(a).

By voltage division,

$$-v_1 = \frac{1}{1+4}(5) = 1 \text{ V}$$

To find V2, we set to zero both the 5-V source and 2 sin 5t current source and transform the circuit to the frequency domain

$$10\cos 2t \quad \Rightarrow \quad 10\angle 0^\circ, \qquad \omega = 2 \text{ rad/s}$$

$$2 \text{ H} \quad \Rightarrow \quad j\omega L = j4\ \Omega$$

$$0.1 \text{ F} \quad \Rightarrow \quad \frac{1}{j\omega C} = -j5\ \Omega$$

Equivalent circuit is shown in Fig. 25(b).

$$\mathbf{Z} = -j5 \parallel 4 = \frac{-j5 \times 4}{4 - j5} = 2.439 - j1.951$$

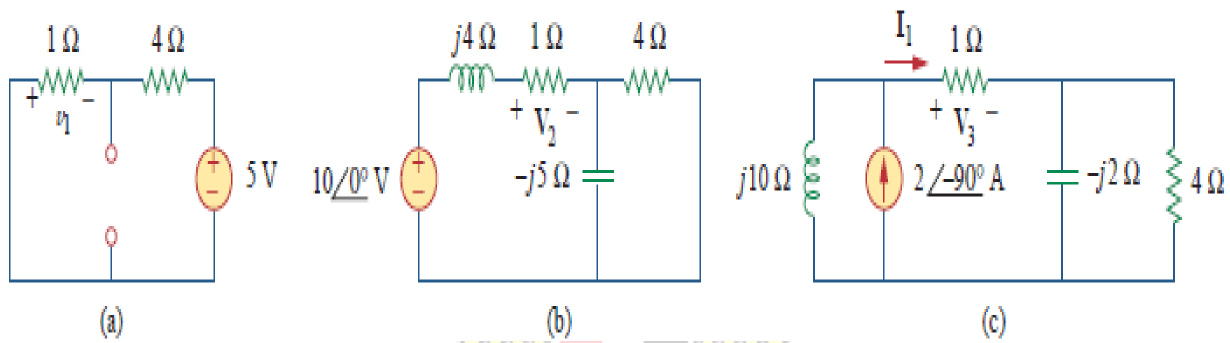


Fig. 25

By voltage division,

$$\mathbf{V}_2 = \frac{1}{1 + j4 + \mathbf{Z}}(10\underline{/0^\circ}) = \frac{10}{3.439 + j2.049} = 2.498\underline{/-30.79^\circ}$$

In time domain

$$v_2 = 2.498\cos(2t - 30.79^\circ)$$

To obtain V3 we set the voltage sources to zero and transform what is left to the frequency domain

$$2\sin 5t \quad \Rightarrow \quad 2\underline{/-90^\circ}, \qquad \omega = 5 \text{ rad/s}$$

$$2\text{ H} \quad \Rightarrow \quad j\omega L = j10\ \Omega$$

$$0.1\text{ F} \quad \Rightarrow \quad \frac{1}{j\omega C} = -j2\ \Omega$$

Equivalent circuit is shown in Fig. 25(c)

$$\mathbf{Z}_1 = -j2 \parallel 4 = \frac{-j2 \times 4}{4 - j2} = 0.8 - j1.6\ \Omega$$

By current divider

$$\mathbf{I}_1 = \frac{j10}{j10 + 1 + \mathbf{Z}_1}(2\underline{/-90^\circ})\text{ A}$$

$$\mathbf{V}_3 = \mathbf{I}_1 \times 1 = \frac{j10}{1.8 + j8.4}(-j2) = 2.328\underline{/-80^\circ}\text{ V}$$

In time domain

$$v_3 = 2.33\cos(5t - 80^\circ) = 2.33\sin(5t + 10^\circ)\text{ V}$$

Thus

$$v_o(t) = -1 + 2.498\cos(2t - 30.79^\circ) + 2.33\sin(5t + 10^\circ)\text{ V}$$

Thevenin Equivalent Circuit

Thevenin's theorem is applied to ac circuits in the same way as they are to dc circuits. The only additional effort arises from the need to manipulate complex numbers. The frequency domain version of a Thevenin equivalent circuit is depicted in Fig. 26, where a linear circuit is replaced by a voltage source in series with an impedance.

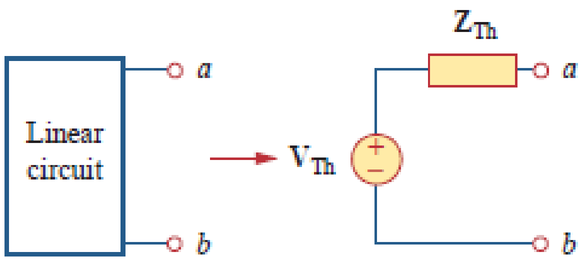


Fig. 26 Thevenin equivalent

Example: Obtain the Thevenin equivalent at terminals a-b of the circuit in Figure 27

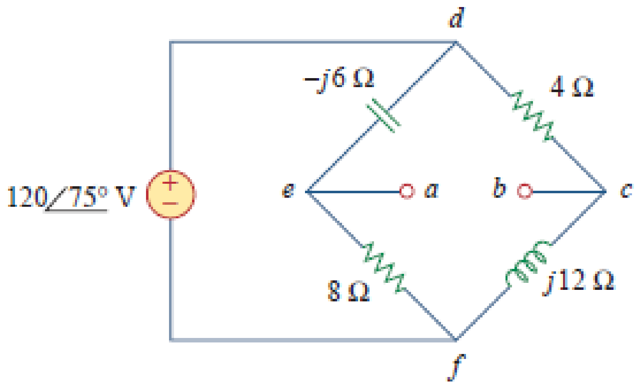


Fig. 27

Solution

We find Z_{TH} by setting the voltage source to zero. As shown in Fig. 28(a), the 8 Ω resistance is now in parallel with the – j 6 reactance, so that their combination gives

$$\mathbf{Z}_1 = -j6 \parallel 8 = \frac{-j6 \times 8}{8 - j6} = 2.88 - j3.84\ \Omega$$

Similarly, the 4 Ω resistance is in parallel with the j12 reactance, and their combination gives

$$\mathbf{Z}_2 = 4 \parallel j12 = \frac{j12 \times 4}{4 + j12} = 3.6 + j1.2\ \Omega$$

The Thevenin impedance is the series combination of Z_1 and Z_2 that is,

$$\mathbf{Z}_{Th} = \mathbf{Z}_1 + \mathbf{Z}_2 = 6.48 - j2.64\ \Omega$$

To find V_{TH} consider the circuit in Fig. 28(b). Currents I_1 and I_2 are obtained as

$$\mathbf{I}_1 = \frac{120\underline{/75^\circ}}{8 - j6}\ \text{A}, \qquad \mathbf{I}_2 = \frac{120\underline{/75^\circ}}{4 + j12}\ \text{A}$$

Applying KVL around loop bcdeab in Fig. 28(b) gives

$$\mathbf{V}_{Th} - 4\mathbf{I}_2 + (-j6)\mathbf{I}_1 = 0$$

$$\begin{aligned}\mathbf{V}_{Th} = 4\mathbf{I}_2 + j6\mathbf{I}_1 &= \frac{480\underline{/75^\circ}}{4 + j12} + \frac{720\underline{/75^\circ + 90^\circ}}{8 - j6} \\ &= 37.95\underline{/3.43^\circ} + 72\underline{/201.87^\circ} \\ &= -28.936 - j24.55 = 37.95\underline{/220.31^\circ}\ \text{V}\end{aligned}$$

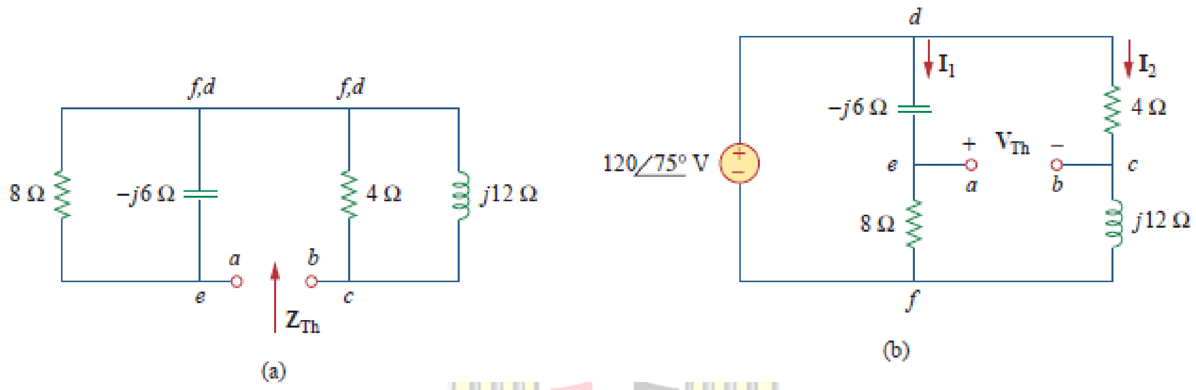


Fig. 28

Example: Find the Thevenin equivalent at terminals a-b of the circuit in Fig. 29

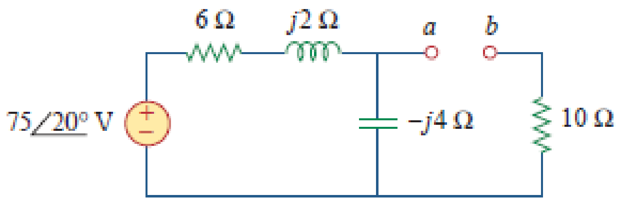


Fig. 29

Answer: $\mathbf{Z}_{Th} = 12.4 - j3.2\ \Omega$, $\mathbf{V}_{Th} = 47.42\underline{/-51.57^\circ}$ V.

Chapter 4

Power electronic switching circuits

Objective

- Overview of the Power electronic control of electric circuits
- Understand the basics of Single-phase uncontrolled rectification circuits
- Understand the basics of Single-phase controlled rectification circuits
- Understand the basics of Single-phase inversion circuits

Power Electronics Backgrounds

The first electronics revolution began in 1948 with the invention of the silicon transistor at Bell Telephone Laboratories by Bardeen, Bratain, and Schockley. Most of today's advanced electronic technologies are traceable to that invention, and modern microelectronics has evolved over the years from these silicon semiconductors. The second electronics revolution began with the development of a commercial thyristor by the General Electric Company in 1958. That was the beginning of a new era of power electronics. Since then, many different types of power semiconductor devices and conversion techniques have been introduced.

The demand for energy, particularly in electrical forms, is ever-increasing in order to improve the standard of living. Power electronics helps with the efficient use of electricity, thereby reducing power consumption. Semiconductor devices are used as switches for power conversion or processing, as are solid state electronics for efficient control of the amount of power and energy flow. Higher efficiency and lower losses are sought for devices used in a range of applications, from microwave ovens to high-voltage dc transmission. New devices and power electronic systems are now evolving for even more effective control of power and energy. Power electronics has already found an important place in modern technology and has revolutionized control of power and energy. As the voltage and current ratings and switching characteristics of power semiconductor devices keep improving, the range of applications continue to expand in areas, such as lamp controls, power supplies to motion control, factory automation, transportation, energy storage, multi-megawatt industrial drives, and electric power transmission and distribution. The greater efficiency and tighter control features of power electronics are becoming attractive for applications in motion control by replacing the earlier electromechanical and electronic systems. Applications in power transmission and renewable energy include high-voltage dc (VHDC) converter stations, flexible ac transmission system (FACTS), static var compensators, and energy storage. In power distribution, these include dc-to-ac conversion, dynamic filters, frequency conversion, and custom power system. Almost all new electrical or electromechanical equipments, from household air conditioners and computer power supplies to industrial motor controls, contain power electronic circuits and/or systems. In order to keep up, working engineers involved in control and conversion of power and energy into applications ranging from several hundred voltages at a fraction of an ampere for display devices to about 10,000 V at high-voltage dc transmission should have a working knowledge of power electronics.

DEFINITION Power electronics involves the study of electronic circuits intended to control the flow of electrical energy. These circuits handle power flow at levels much higher than the individual device ratings. Rectifiers are probably the most familiar examples of circuits that meet this definition. Inverters (a general term for dc-ac converters) and dc-dc converters for power supplies are also common applications. As shown in Fig. 1, power electronics represents a median point at which the topics of energy systems, electronics, and control converge and combine. Any useful circuit design for an energy application must address issues of both devices and control, as well as of the energy itself. Among the unique aspects of power electronics are its emphasis on large semiconductor devices, the application of magnetic devices for energy storage, special control methods that must be applied to nonlinear systems, and its fundamental place as a central component of today's energy systems and alternative resources. In any study of electrical engineering, power electronics must be placed on a level with digital, analog, and radio-frequency electronics to reflect the distinctive design methods and unique challenges.

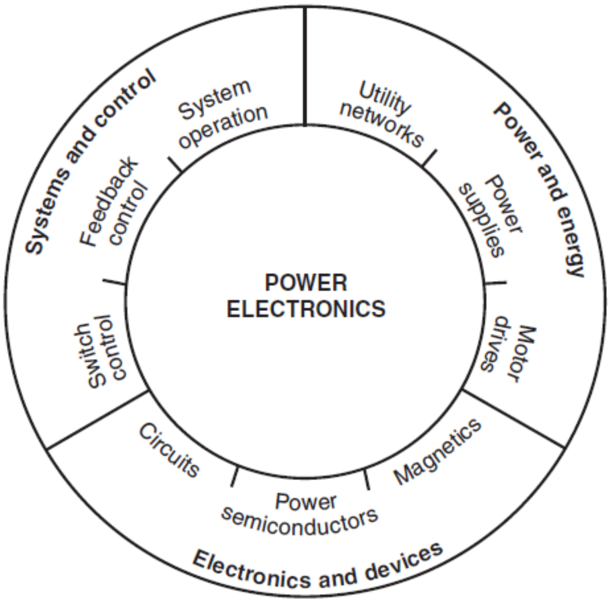


Fig. 1 Control, energy and power electronics are interrelated

The history of power electronics has been closely allied with advances in electronic devices that provide the capability to handle high power levels. Since about 1990, devices have become so capable that a transition from a "device-driven" field to an "applications-driven" field continues. This transition has been based on two factors: (1) advanced semiconductors with suitable power ratings exist for almost every application of wide interest, and (2) the general push toward miniaturization is bringing advanced power electronics into a growing variety of products. Although the devices continue to improve, their development now tends to follow innovative applications.

Key Characteristics

All power electronic circuits manage the flow of electrical energy between an electrical source and a load. The parts in a circuit must direct electrical flows, not impede them. A general power conversion system is shown in Fig. 2. The function of the power converter in the middle is to control the energy flow between a source and a load. For our purposes, the power converter will be implemented with a power electronic circuit. Because a power converter appears between a source and a load, any energy used within the converter is lost to the overall system. A crucial point emerges: to build a power converter, we should consider only lossless components. A realistic converter design must approach 100% efficiency. A power converter connected between a source and a load also affects system reliability. If the energy source is perfectly reliable (it is available all the time), then a failure in the converter affects the user (the load) just as if the energy source had failed. An unreliable power converter creates an unreliable system. To put this in perspective, consider that a typical American household loses electric power only a few minutes a year. Energy is available 99.999% of the time. A converter must be better than this to prevent system degradation. An ideal converter implementation will not suffer any failures over its application lifetime. Extreme high reliability can be a more difficult objective than high efficiency.

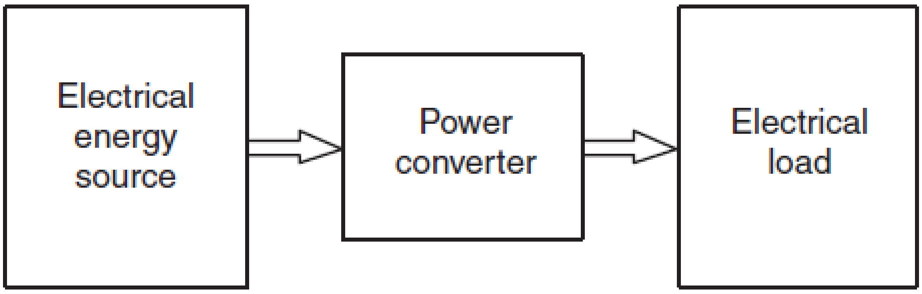


Fig. 2 General system for electric power conversion

Switch principles

A circuit element as simple as a light switch reminds us that the extreme requirements in power electronics are not especially novel. Ideally, when a switch is on, it has zero voltage drop and will carry any current imposed on it. When a switch is off, it blocks the flow of current regardless of the voltage across it. The device power, the product of the switch voltage and current, is identically zero at all times. A switch therefore controls energy flow with no loss. In addition, reliability is also high. Household light switches perform over decades of use and perhaps 100,000 operations. Unfortunately, a mechanical light switch does not meet all practical needs. A switch in a power supply may function 100,000 times each second. Even the best mechanical switch will not last beyond a few million cycles. Semiconductor switches (without this limitation) are the devices of choice in power converters. A circuit built from ideal switches will be lossless. As a result, switches are the main components of power converters, and many people equate power electronics with the study of switching power converters. Magnetic transformers and lossless storage elements such as capacitors and inductors are also valid components for use in power converters. The complete concept, shown in Fig. 3, illustrates a power electronic system. Such a system consists of an electrical energy source, an electrical load, a power electronic circuit, and a control function. The power electronic circuit contains switches, lossless energy storage elements, and magnetic transformers. The controls take information from the source, the load, and the designer, and then determine how the switches operate to achieve the desired conversion. The controls are built up with low-power analog and digital electronics.

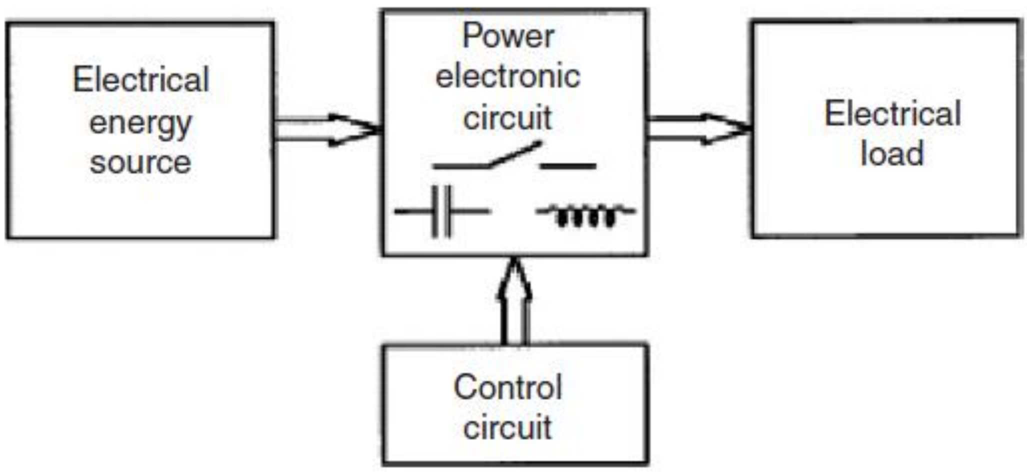


Fig. 3 A basic power electronic system

Controlled rectification circuit

In this section, the uncontrolled rectification circuit will be studied. It mainly consists of Diodes act as switches to convert from AC to DC.

Diode as a Switch

Among all the static switching devices used in power electronics (PE), the power diode is perhaps the simplest. Its circuit symbol is shown in Fig. 4. It is a two terminal device, and terminal A is known as the anode whereas terminal K is known as the cathode. If terminal A experiences a higher potential compared to terminal K, the device is said to be forward biased and a current called forward current (IF) will flow through the device in the direction as shown. This causes a small voltage drop across the device (<1V), which in ideal condition is usually ignored. On the contrary, when a diode is reverse biased, it does not conduct and a practical diode does experience a small current flowing in the reverse direction called the leakage current. Both the forward voltage drop and the leakage current are ignored in an ideal diode. Usually in PE applications a diode is considered to be an ideal static switch. The characteristics of a practical diode show a departure from the ideals of zero forward and infinite reverse impedance, as shown in Fig. 5-a. In the forward direction, a potential barrier associated with the distribution of charges in the vicinity of the junction, together with other effects, leads to a voltage drop. This, in the case of silicon, is in the range of 1V for currents in the normal range. In reverse, within the normal operating range of voltage, a very small current flow which is largely independent of the voltage. For practical purposes, the static characteristic is often represented by Fig. 5-b. In the Figure, the forward characteristic is expressed as a threshold voltage V_o and a linear incremental or slope resistance, r.

The reverse characteristic remains the same over the range of possible leakage currents irrespective of voltage within the normal working range.

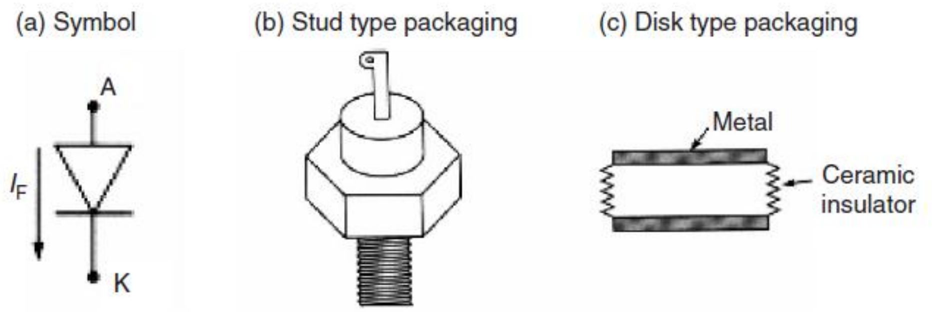


Fig. 4 power diode (a) symbol; (b) and (c) types of packing

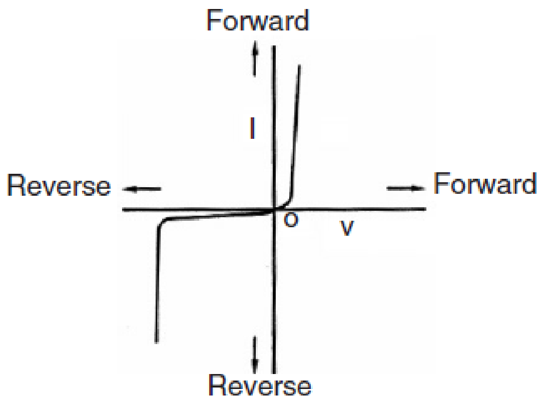


Fig. 5-a Typical static c/c's of a power diode

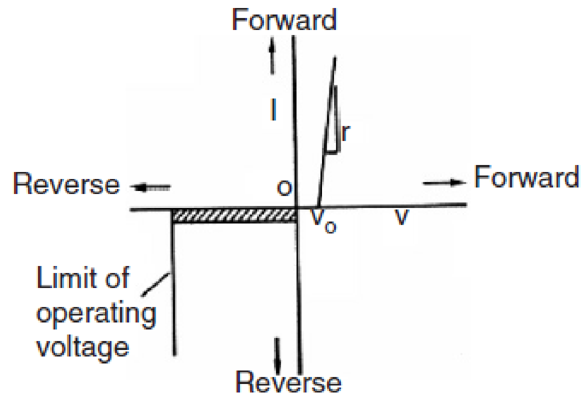


Fig. 5-b Practical representation of the static c/c's of a power diode

Properties of PN Junction

From the forward and reverse biased condition characteristics, one can notice that when the diode is forward biased, current rises rapidly as the voltage is increased. Current in the reverse biased region is significantly small until the breakdown voltage of the diode is reached. Once the applied voltage is over this limit, the current will increase rapidly to a very high value limited only by an external resistance.

DC diode parameters:

The most important parameters are the followings:

- Forward voltage, V_F is the voltage drop of a diode across A and K at a defined current level when it is forward biased.
- Breakdown voltage, V_B is the voltage drop across the diode at a defined current level when it is beyond reverse biased level. This is popularly known as avalanche.
- Reverse current I_R is the current at a particular voltage, which is below the breakdown voltage.

Single-phase uncontrolled rectification circuits

Electrical energy sources take the form of dc voltage sources at various values, sinusoidal ac sources, polyphase sources, among others. A power electronic circuit might be asked to transfer energy between two different dc voltage levels, between an AC source and a dc load, or between sources at different frequencies. It might be used to adjust an output voltage or power level, drive a nonlinear load, or control a load current. In this section, a few basic converter arrangements are introduced, and energy conservation provides a tool for analysis.

EXAMPLE: Consider the circuit shown in Fig. 6. It contains an AC source, a switch, and a resistive load. It is a simple but complete power electronic system.

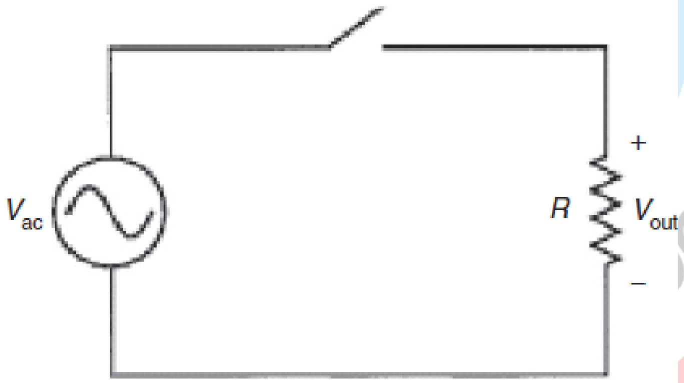


Fig. 6 A simple power electronic system

Let us assign a (somewhat arbitrary) control scheme to the switch. What if the switch is turned on whenever Vac >0, and turned off otherwise? The input and output voltage waveforms are shown in Fig. 7. The input has a time average of 0, and root- mean-square (RMS) value equal to $V_{peak}/\sqrt{2}$, where V_{peak} is the maximum value of Vac. The output has a nonzero average value given by

$$V_{out}(t) = (1/2\pi)\left(\int_{-\pi/2}^{\pi/2} Vpeak\cos\theta\, d\theta + \int_{\pi/2}^{3\pi/2} 0\, d\theta\right)$$

$$= V_{peak} / \pi = 0.3183\, V_{peak} \qquad (1)$$

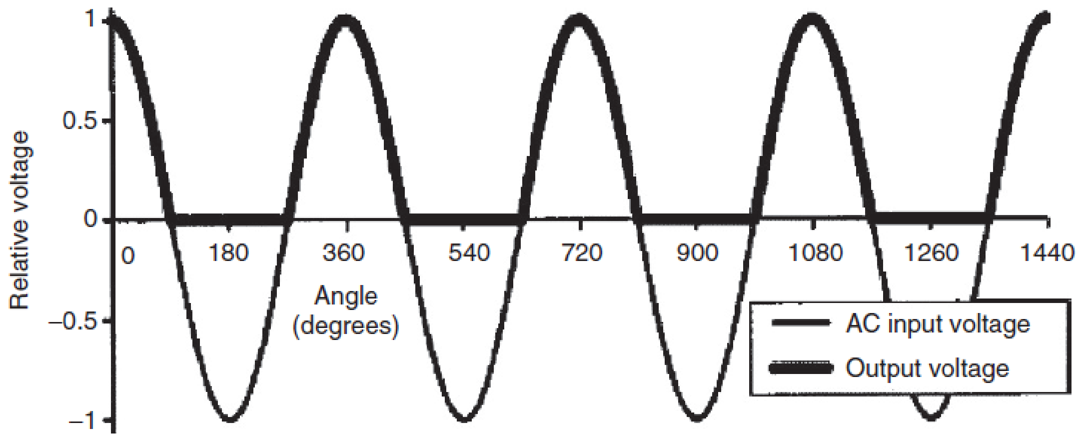


Fig. 7 Input and output waveforms for the Example

and an RMS value equal to $V_{peak}/2$. Since the output has nonzero dc voltage content, the circuit can be used as an ac-dc converter. To make it more useful, a low-pass filter would be added between the output and the load to smooth out the ac portion. This filters needs to be lossless, and will be constructed from only inductors and capacitors. The circuit in the previous Example acts as a half-wave uncontrolled rectifier with a resistive load. With the hypothesized switch action, a diode (uncontrolled switch) can substitute for the ideal switch. The example confirms that a simple switching circuit can perform power conversion functions. But note that a diode is not, in general, the same as an ideal switch. A diode places restrictions on the current direction, whereas a true switch would not. An ideal switch allows control over whether it is on or off, whereas a diode's operation is constrained by circuit variables. Consider a second half-wave circuit, now with a series L–R load, shown in Fig. 8.

EXAMPLE: A series diode L–R circuit has ac voltage source input. This circuit operates much differently than the half-wave rectifier with resistive load. A diode will be on if forward-biased, and off if reverse-biased. In this circuit, when the diode is off, the current will be zero.

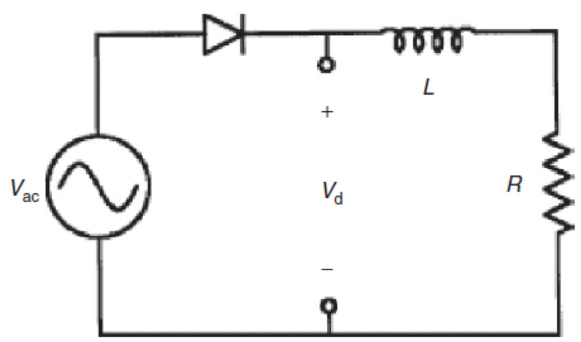


Fig. 8 Half wave rectifier with L-R load

Whenever the diode is on, the circuit is the ac source with L–R load. Let the ac voltage be $V_0 \cos(\omega t)$. From Kirchhoff's Voltage Law (KVL),

$$V_0 \cos(\omega t) = L\, di/dt + R_i \qquad (2)$$

Let us assume that the diode is initially off (this assumption is arbitrary, and we will check it as the example is solved). If the diode is off, the diode current is i =0, and the voltage across the diode will be V_{ac}. The diode will become forward-biased when V_{ac} becomes positive. The diode will turn on when the input voltage makes a zero-crossing in the positive direction. This allows us to establish initial conditions for the circuit: $i(t_0) = 0$, $t_0 = -\pi/(2\omega)$. The differential equation can be solved in a conventional way to give

$$i(t) = V_0 \left[\frac{\omega L}{R^2+\omega^2 L^2} * \exp\left(\frac{-t}{\tau} - \frac{\pi}{2\omega\tau}\right) + \frac{R}{R^2+\omega^2 L^2}\cos(\omega t) + \frac{\omega L}{R^2+\omega^2 L^2}\sin(\omega t)\right] \quad (3)$$

Where τ is the time constant L/R.

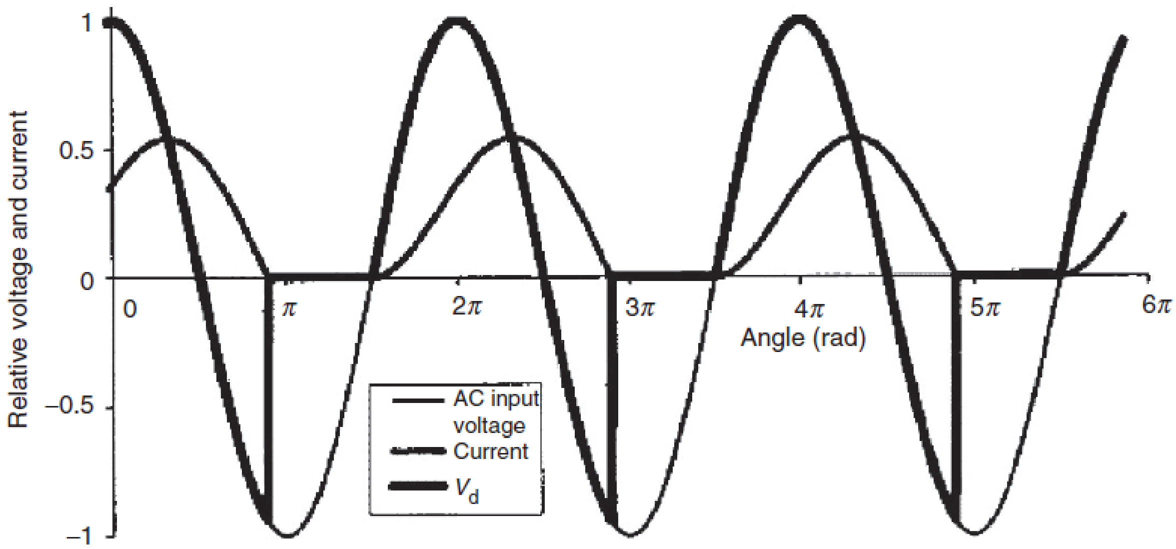


Fig. 9 Input and output waveforms for the previous example

What about when the diode is turned off? The first guess might be that the diode turns off when the voltage becomes negative, which is not correct. From the solution, we can note that the current is not zero when the voltage first becomes negative. If the switch attempts to turn off, it must instantly drop the inductor current to zero. The derivative of current in the inductor, di/dt, would become negative infinite. The inductor voltage L(di/dt) similarly becomes negative infinite, and the devices are destroyed. What really happens is that the falling current allows the inductor to maintain forward bias on the diode. The diode will turn off only when the current reaches zero. A diode has definite properties that determine the circuit action, and both the voltage and current are relevant. Figure 1.7 shows the input and output waveforms for a time constant τ equal to about one-third of the ac waveform period.

Full bridge rectifier

In rectification four diodes can be used to fully rectify an Ac signal as shown in Fig. 10. Apart from other rectifier circuits, this topology does not require an input transformer. However, they are used for isolation and protection. The direction of the current is decided by two diodes conducting at any given time. The direction of the current through the load is always the same. This rectifier topology is known as the full bridge rectifier.

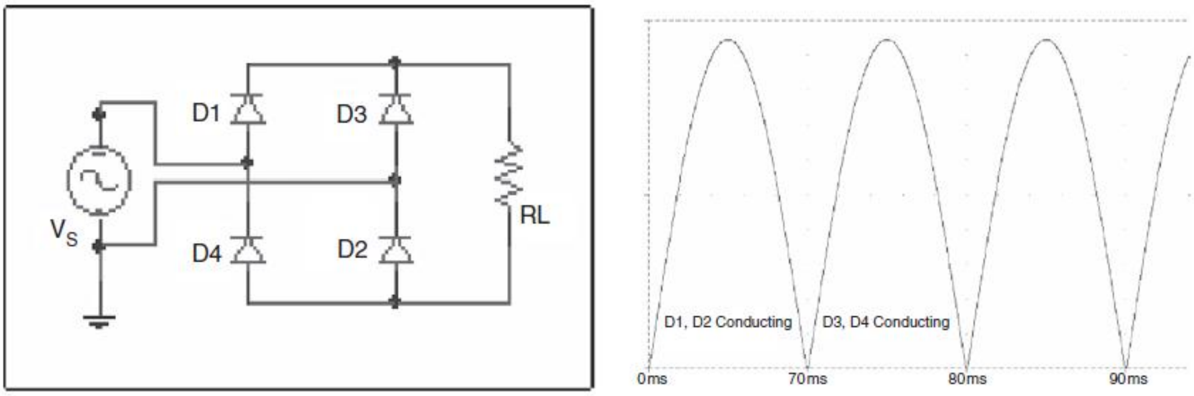


Fig. 10 Full bridge rectifier and its output voltage

The average rectifier output voltage:

$V_{dc} = 2V_m/\pi$ (4)

where V_m is the peak input voltage.

The rms rectifier output voltage:

$V_{rms} = V_m / \sqrt{2}$ (5)

Single-phase controlled rectification circuits

Thyristors are usually three-terminal devices that have four layers of alternating p-type and n-type material (i.e. three p–n junctions) comprising its main power handling section. The control terminal of the thyristor, called the gate (G) electrode, may be connected to an integrated and complex structure as a part of the device. The other two terminals, called the anode (A) and cathode (K), handle the large applied potentials (often of both polarities) and conduct the major current through the thyristor. The anode and cathode terminals are connected in series with the load to which power is to be controlled. Thyristors are used to approximate ideal closed (no voltage drop between anode and cathode) or open (no anode current flow) switches for control of power flow in a circuit. This differs from low-level digital switching circuits that are designed to deliver two distinct small voltage levels while conducting small currents (ideally zero). Thyristor circuits must have the capability of delivering large currents and be able to withstand large externally applied voltages. All thyristor types are controllable in switching from a forward-blocking state (positive potential applied to the anode with respect to the cathode, with correspondingly little anode current flow) into a forward-conduction state (large forward anode current flowing, with a small anode–cathode potential drop). Most thyristors have the characteristic that after switching from a forward-blocking state into the forward-conduction state, the gate signal can be removed and the thyristor will remain in its forward-conduction mode. This property is termed “latching” and is an important distinction between thyristors and other types of power electronic devices. Some thyristors are also controllable in switching from forward-conduction back to a forward-blocking state. The particular design of a thyristor will determine its controllability and often its application. Thyristors are typically used at the highest energy levels in power conditioning circuits because they are designed to handle the largest currents and voltages of any device technology (systems approximately with voltages above 1 kV or currents above 100 A).

A thyristor used in some ac power circuits (50 or 60 Hz in commercial utilities or 400 Hz in aircraft) to control ac power flow can be made to optimize internal power loss at the expense of switching speed. These thyristors are called phase-control devices, because they are generally turned from a forward blocking into a forward-conducting state at some specified phase angle of the applied sinusoidal anode-cathode voltage waveform. A second class of thyristors is used in association with dc sources or in converting ac power at one amplitude and frequency into ac power at another amplitude and frequency, and must generally switch on and off relatively quickly. A typical application for the second class of thyristors is in converting a dc voltage or current into an Ac voltage or current. A circuit that performs this operation is often called an inverter, and the associated thyristors used are referred to as inverter thyristors. Figure 11 shows a conceptual view of a typical thyristor with the three p-n junctions and the external electrodes labeled. Also shown in the Figure is the thyristor circuit symbol used in electrical schematics. Figure 12 shows typical thyristor stud-mount and press-pack packages.

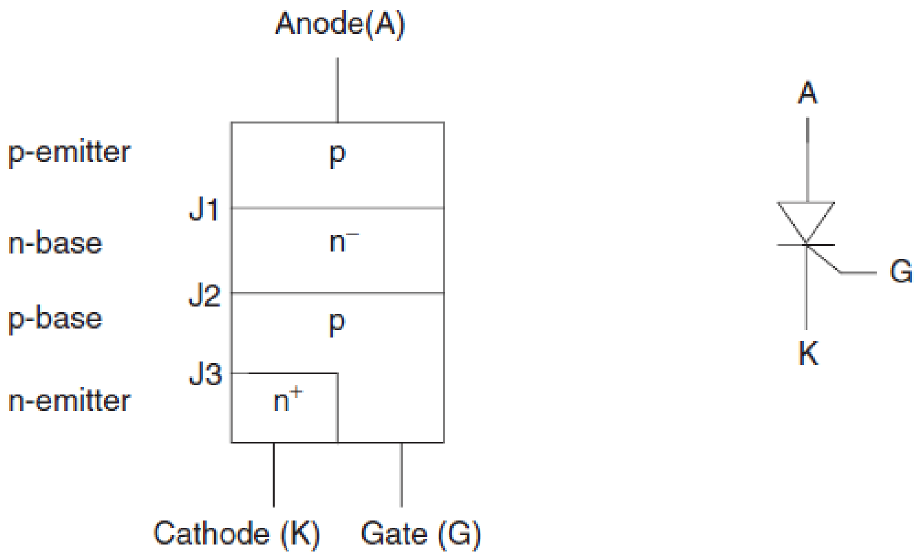


Fig 11 Simple cross section of a thyristor and its electrical symbol

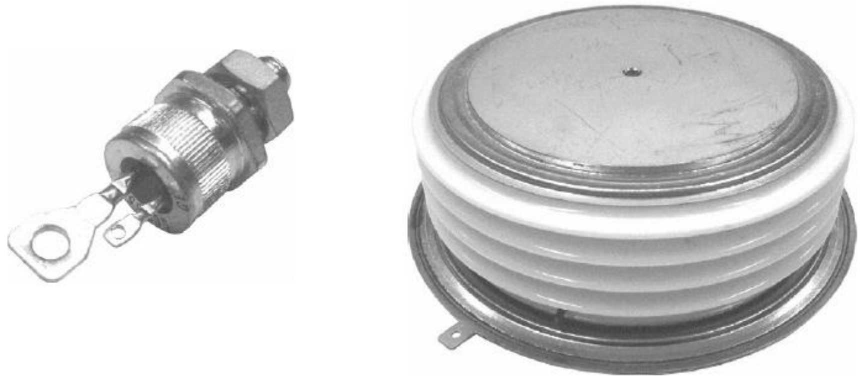


Figure 12 typical thyristor stud-mount and press-pack packages.

Applications

The most important application of thyristors is for line frequency phase-controlled rectifiers. This family includes several topologies, of which one of the most important is used to construct HVDC transmission systems. A single-phase controlled rectifier is shown in Fig. 13. The use of thyristors instead of diodes allows the average output voltage to be controlled by appropriate gating of the thyristors. If the gate signals to the thyristors were continuously applied, the thyristors in Fig. 6.13 would behave as diodes. If no gate currents are supplied they behave as open circuits. Gate current can be applied any time (phase delay) after the forward voltage becomes positive. Using this phase-control feature, it is possible to produce an average dc output voltage less than the average output voltage obtained from an uncontrolled diode rectifier.

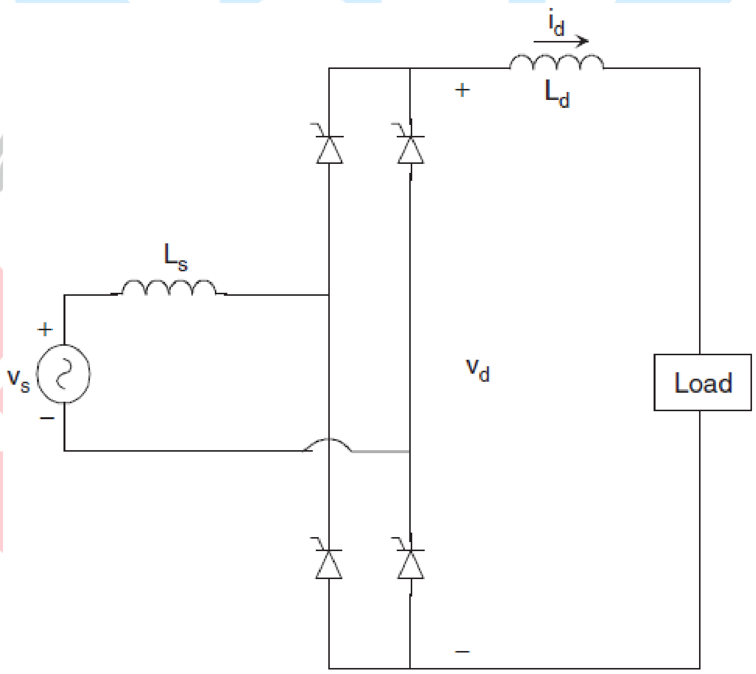


Fig. 13 Single phase controlled rectifier circuit

Single-phase controlled Half-wave Rectifier

The single-phase half-wave rectifier uses a single thyristor to control the load voltage as shown in Fig. 14. The thyristor will conduct, on-state, when the voltage v_T is positive and a firing current pulse i_G is applied to the gate terminal. The control of the load voltage is performed by delaying the firing pulse by an angle α. The firing angle α is measured from the position where a diode would naturally conduct. In case of Fig. 14, the angle α is measured from the zero-crossing point of the supply voltage v_s. The load in Fig. 14 is resistive and therefore the current id has the same waveform of the load voltage. The thyristor goes to the non-conducting condition, off-state, when the load voltage, and consequently the current, reaches a negative value. The load average voltage is given by

$$V_{d\alpha} = \frac{1}{2\pi}\int_{\alpha}^{\pi} V_{max}\sin(\omega t)d(\omega t) = \frac{V_{max}}{2\pi}(1+\cos\alpha) \qquad (6)$$

Where V_{max} is the supply peak voltage. Hence, it can be seen from Eq. (6) that changing the firing angle α controls both the load average voltage and the amount of transferred power. Figure 15-a shows the rectifier waveforms for an R-L load.

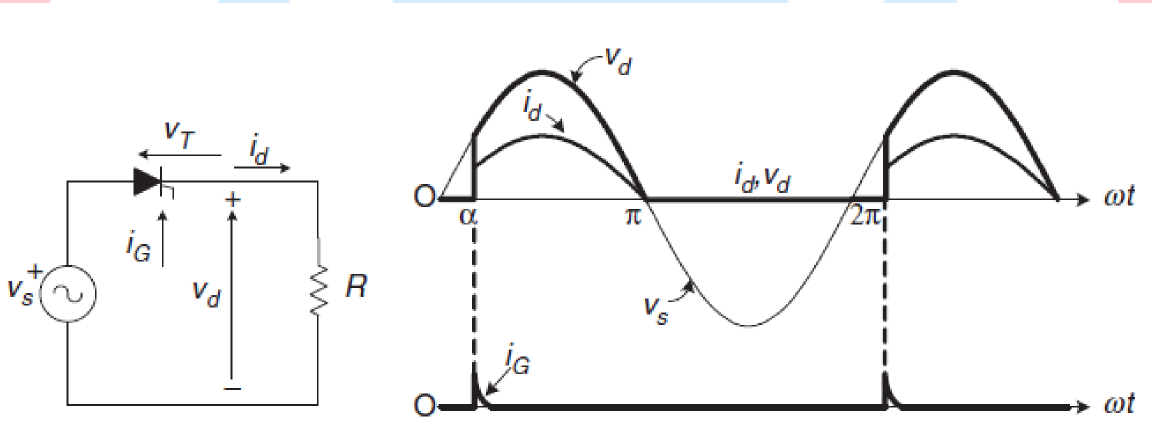


Fig. 14 single thyristor rectifier with resistive load

When the thyristor is turned on, the voltage across the inductance is

$$V_L = V_s - V_R = L\, di_d/dt \qquad (7)$$

Where V_R is the voltage in the resistance R, given by $V_R = R*i_d$. If $V_s - V_R > 0$, from Eq. (7) holds that the load current increases its value. On the other hand, i_d decreases its value when $V_s - V_R < 0$. The load current is given by

$$i_d(\omega t) = \frac{1}{\omega L}\int_{\alpha}^{\omega t} V_L \, d\theta \qquad (8)$$

Graphically, Eq. (8) means that the load current i_d is equal to zero when $A_1 = A_2$, maintaining the thyristor in conduction state even when $V_s < 0$. When an inductive active load is connected to the rectifier, as illustrated in Fig. 15-b, the thyristor will be turned on if the firing pulse is applied to the gate when $V_s > E_d$. Again, the thyristor will remain in the on-state until $A_1 = A_2$. When the thyristor is turned off, the load voltage will be $V_d = E_d$.

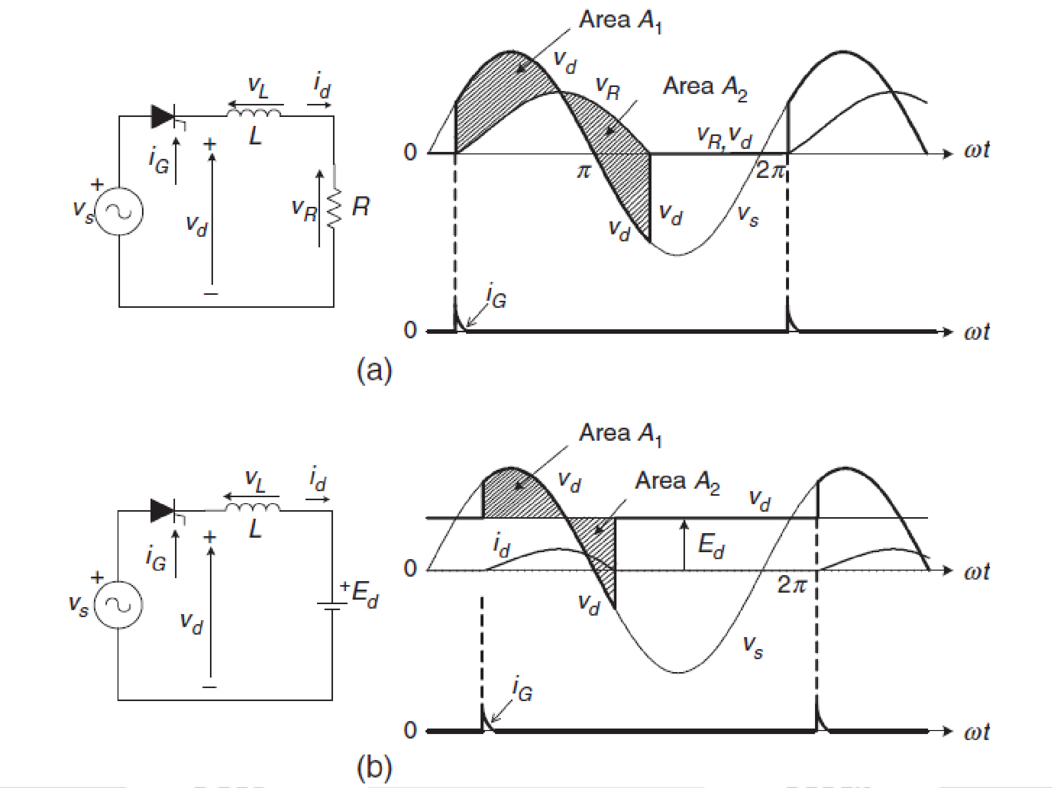


Fig. 15 single thyristor rectifier with: (a) R-L load (b) active load

Single-phase controlled Bridge Rectifier

Figure 16 shows a fully controlled bridge rectifier, which uses four thyristors to control the average load voltage. In addition, Fig. 16-b shows the half-controlled bridge rectifier which uses two thyristors and two diodes. The voltage and current waveforms of the fully controlled bridge rectifier for a resistive load are illustrated in Fig 17. Thyristors T_1 and T_2 must be fired on simultaneously during the positive half-wave of the source voltage V_s, to allow the conduction of current. Alternatively, thyristors T_3 and T_4 must be fired simultaneously during the negative half-wave of the source voltage. To ensure simultaneous firing, thyristors T_1 and T_2 use the same firing signal. The load voltage is similar to the voltage obtained with the bi-phase half-wave rectifier. The input current is given by

$i_s = i_{T1} - i_{T4}$ (9)

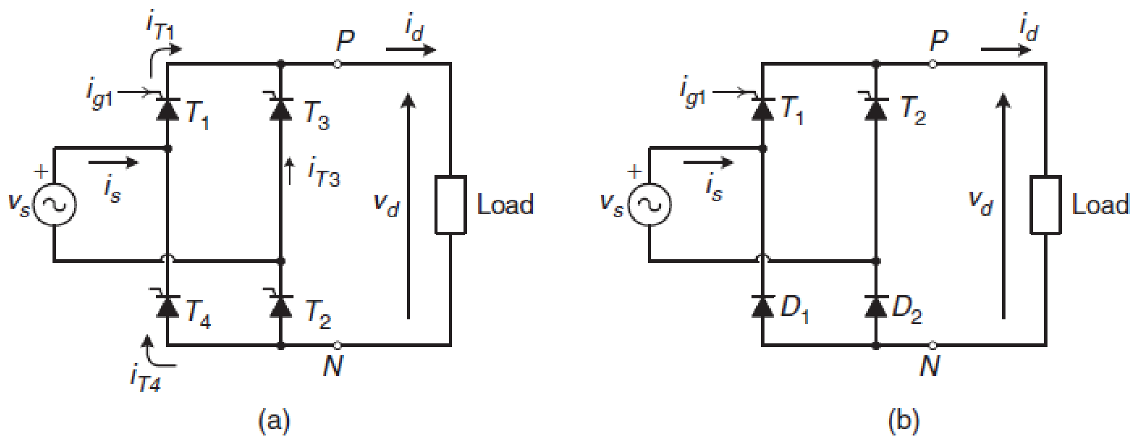


Fig. 16 Single phase bridge rectifier (a) fully controlled (b) half controlled

and its waveform is shown in Fig. 17. Figure 18 presents the behavior of the fully controlled rectifier with resistive-inductive load (with $L \rightarrow \infty$). The high load inductance generates a perfectly filtered current and the rectifier behaves like a current source. With continuous load current, thyristors T_1 and T_2 remain in the on-state beyond the positive half-wave of the source voltage V_s . For this reason, the load voltage V_d can have a negative instantaneous value. The firing of thyristors T_3 and T_4 has two effects:

(i) They turn-off thyristors T_1 and T_2

(ii) After the commutation, they conduct the load current.

This is the main reason why this type of converters are called “naturally commutated” or “line commutated” rectifiers. The supply current i_S has the square waveform, as shown in Fig. 18, for continuous conduction. In this case, the average load voltage is given by

$$V_{d\alpha} = \frac{1}{\pi}\int_{\alpha}^{\pi+\alpha} V_{max}\,\sin(\omega t)\,d\,(\omega t) = \frac{2V_{max}}{\pi}\cos\alpha \qquad (10)$$

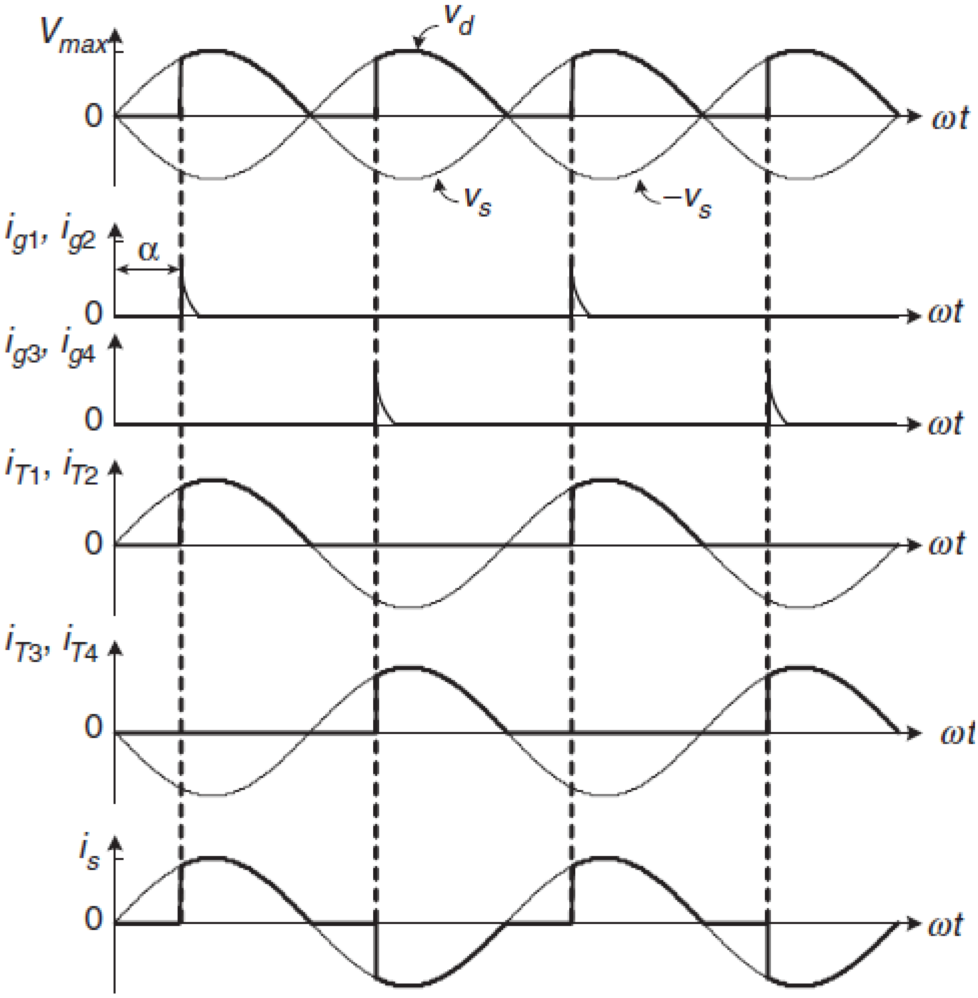


Fig. 17 Waveforms of a fully controlled bridge rectifier with R-load

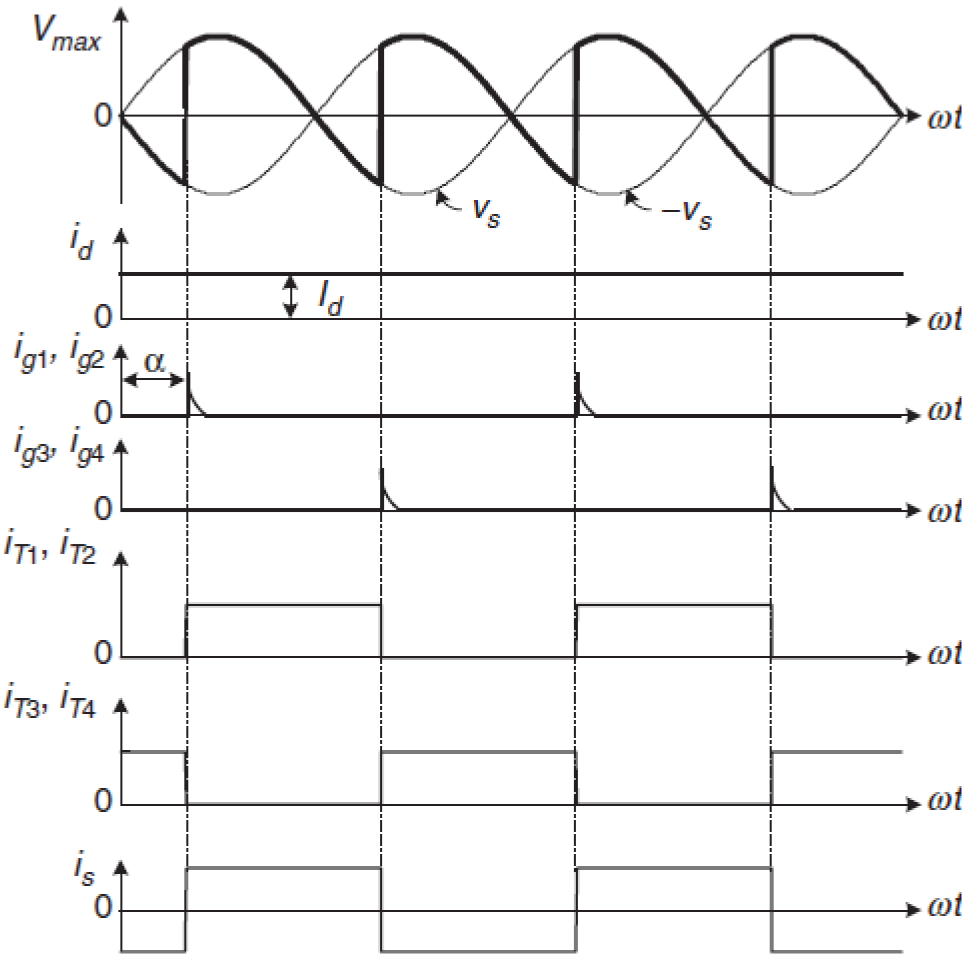


Fig. 18 Waveforms of a fully controlled bridge rectifier with R-L load (L→ ∞)

Single-phase inversion circuits

The main objective of static power converters is to produce an Ac output waveform from a dc power supply. These are the types of waveforms required in adjustable speed drives (ASDs), uninterruptible power supplies (UPSs), static var compensators, active filters, flexible ac transmission systems (FACTSs), and voltage compensators, which are only a few applications. For sinusoidal Ac outputs, the magnitude, frequency, and phase should be controllable. According to the type of ac output waveform, these topologies can be considered as voltage-source inverters (VSIs), where the independently controlled ac output is a voltage waveform. These structures are the most widely used because they naturally behave as voltage sources as required by many industrial applications, such as ASDs, which are the most popular application of inverters (Fig. 19-a). Similarly, these topologies can be found as current-source inverters (CSIs), where the independently controlled Ac output is a current waveform. These structures are still widely used in medium-voltage industrial applications, where high-quality voltage waveforms are required. Static power converters, specifically inverters, are constructed from power switches and the Ac output waveforms are therefore made up of discrete values. This leads to the generation of waveforms that feature fast transitions rather than smooth ones. For instance, the Ac output voltage produced by the VSI of a three-level ASD is a, Pulse Width Modulation (PWM) type of waveform (Fig. 19). Although this waveform is not sinusoidal as expected (Fig. 19), its fundamental component behaves as such. This behavior should be ensured by a modulating technique that controls the amount of time and the sequence used to switch the power valves on and off.

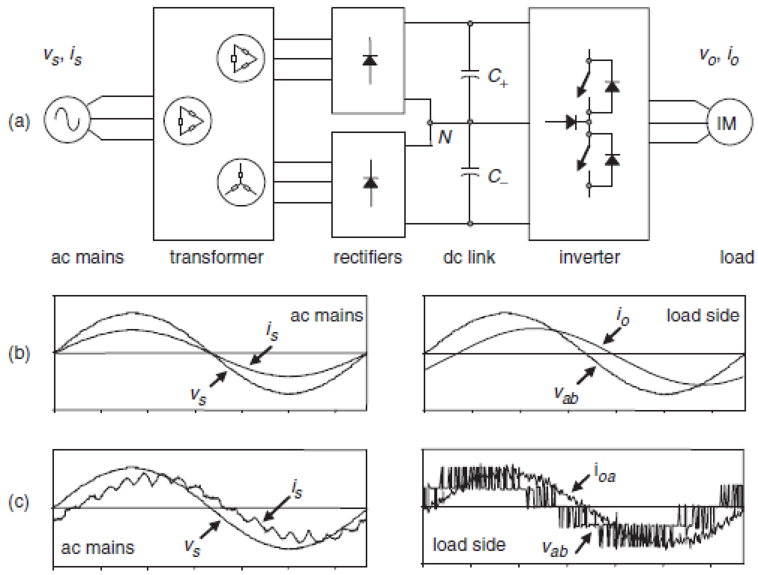


Fig. 19 A three level adjustable speed drive scheme and associated waveforms (a) power conversion topology (b) ideal input and output waveforms (c) actual input and output

waveforms

The modulating techniques most used are the carrier-based technique (e.g. sinusoidal pulse width modulation, SPWM), the space-vector (SV) technique, and the selective-harmonic-elimination (SHE) technique.

The discrete shape of the ac output waveforms generated by these topologies imposes basic restrictions on the applications of inverters. The VSI generates an Ac output voltage waveform composed of discrete values (high dv/dt); therefore, the load should be inductive at the harmonic frequencies in order to produce a smooth current waveform. A capacitive load in the VSIs will generate large current spikes. If this is the case, an inductive filter between the VSI Ac side and the load should be used. On the other hand, the CSI generates an Ac output current waveform composed of discrete values (high di/dt); therefore, the load should be capacitive at the harmonic frequencies in order to produce a smooth voltage waveform. An inductive load in CSIs will generate large voltage spikes. If this is the case, a capacitive filter between the CSI Ac side and the load should be used. A three-level voltage waveform is not recommended for medium-voltage ASDs due to the high dv/dt that would apply to the motor terminals. Several negative side effects of this approach have been reported (bearing and isolation problems). As alternatives, to improve the ac output waveforms

in VSIs are the multistage topologies (multilevel and multi-cell). The basic principle is to construct the required Ac output waveform from various voltage levels, which achieves medium-voltage waveforms at reduced dv/dt . Although these topologies are well developed in ASDs, they are also suitable for static var compensators, active filters, and voltage compensators. Specialized modulating techniques have been developed to switch the higher number of power valves involved in these topologies. Among others, the carrier-based (SPWM) and SV-based techniques have been naturally extended to these applications.

In many applications, it is required to take energy from the ac side of the inverter and send it back into the dc side. For instance, whenever ASDs need to either brake or slow down the motor speed, the kinetic energy is sent into the voltage dc link (Fig. 19-a). This is known as the regenerative operating mode and, in contrast to the motoring mode, the dc link current direction is reversed due to the fact that the dc link voltage is fixed. If a capacitor is used to maintain the dc link voltage (as in standard ASDs) the energy must either be dissipated or fed back into the distribution system, otherwise, the dc link voltage gradually increases. The first approach requires the dc link capacitor be connected in parallel with a resistor, which must be properly switched only when the energy flows from the motor into the dc link. A better alternative is to feed back such energy into the distribution system. However, this alternative requires a reversible-current topology connected between the distribution system and the dc link capacitor. A modern approach to such a requirement is to use the active front-end rectifier technologies, where the regeneration mode is a natural operating mode of the system.

Single-phase Voltage Source Inverters

Single-phase VSI can be found as half-bridge and full-bridge topologies. Although, the power range they cover is the low one, they are widely used in power supplies, single-phase UPSs, and currently to form high-power static power topologies, the main features of both approaches are reviewed and presented in the following.

Half-bridge VSI

Figure 20 shows the power topology of a half-bridge VSI, where two large capacitors are required to provide a neutral point N, such that each capacitor maintains a constant voltage V_i /2. Because the current harmonics injected by the operation of the inverter are low-order harmonics, a set of large capacitors (C+ and C−) is required. It is clear that both switches S+ and S− cannot be on simultaneously because a short circuit across the dc link voltage source V_i would be produced. There are two defined (states 1 and 2) and one undefined (state 3) switch state as shown in Table 1. In order to avoid the short circuit across the dc bus and the undefined Ac output-voltage condition, the modulating technique should always ensure that at any instant either the top or the bottom switch of the inverter leg is on.

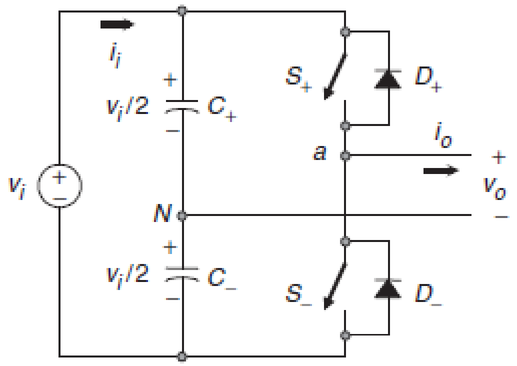


Fig. 20 Single phase half bridge VSI

Table 1 Switch states for Single phase half bridge VSI

State	State #	v_o	Components conducting
S_+ is on and S_- is off	1	$v_i/2$	S_+ if $i_o > 0$ D_+ if $i_o < 0$
S_- is on and S_+ is off	2	$-v_i/2$	D_- if $i_o > 0$ S_- if $i_o < 0$
S_+ and S_- are all off	3	$-v_i/2$ $v_i/2$	D_- if $i_o > 0$ D_+ if $i_o < 0$

Figure 21 shows the ideal waveforms associated with the half-bridge inverter shown in Fig. 20. The states for the switches S+ and S− are defined by the modulating technique, which in this case is a carrier-based PWM.

The Carrier-based Pulse Width Modulation (PWM) Technique

As mentioned earlier, it is desired that the Ac output voltage, $V_o = V_{aN}$, follow a given waveform (e.g. sinusoidal) on a continuous basis by properly switching the power valves. The carrier-based PWM technique fulfills such a requirement as it defines the on- and off-states of the switches of one leg of a VSI by comparing a modulating signal V_c (desired Ac output voltage) and a triangular waveform V_Δ (carrier signal). In practice, when $V_c > V_\Delta$ the switch S+ is on and the switch S− is off; similarly, when $V_c < V_\Delta$ the switch S+ is off and the switch S− is on.

A special case is when the modulating signal V_c is a sinusoidal at frequency f_c and amplitude $\dot{V}_c$, and the triangular signal v_Δ is at frequency f_Δ and amplitude $\dot{V}_\Delta$. This is the sinusoidal PWM (SPWM) scheme. In this case, the modulation index m_a (also known as the amplitude-modulation ratio) Is defined as

$$ma = \dot{V}_C / \dot{V}_\Delta \quad (11)$$

and the normalized carrier frequency m_f (also known as the frequency-modulation ratio) is

$$m_f = f_\Delta / f_c \quad (12)$$

Figure 21-e clearly shows that the Ac output voltage $V_o = V_{aN}$ is basically a sinusoidal waveform plus harmonics, which features: (a) the amplitude of the fundamental component of the Ac output voltage $\dot{V}_{o1}$ satisfying the following expression:

$$\dot{V}_{o1} = \dot{V}_{aN1} = \frac{V_i}{2} m_a \quad (13)$$

for $m_a \leq 1$, which is called the linear region of the modulating technique (higher values of ma leads to over-modulation)

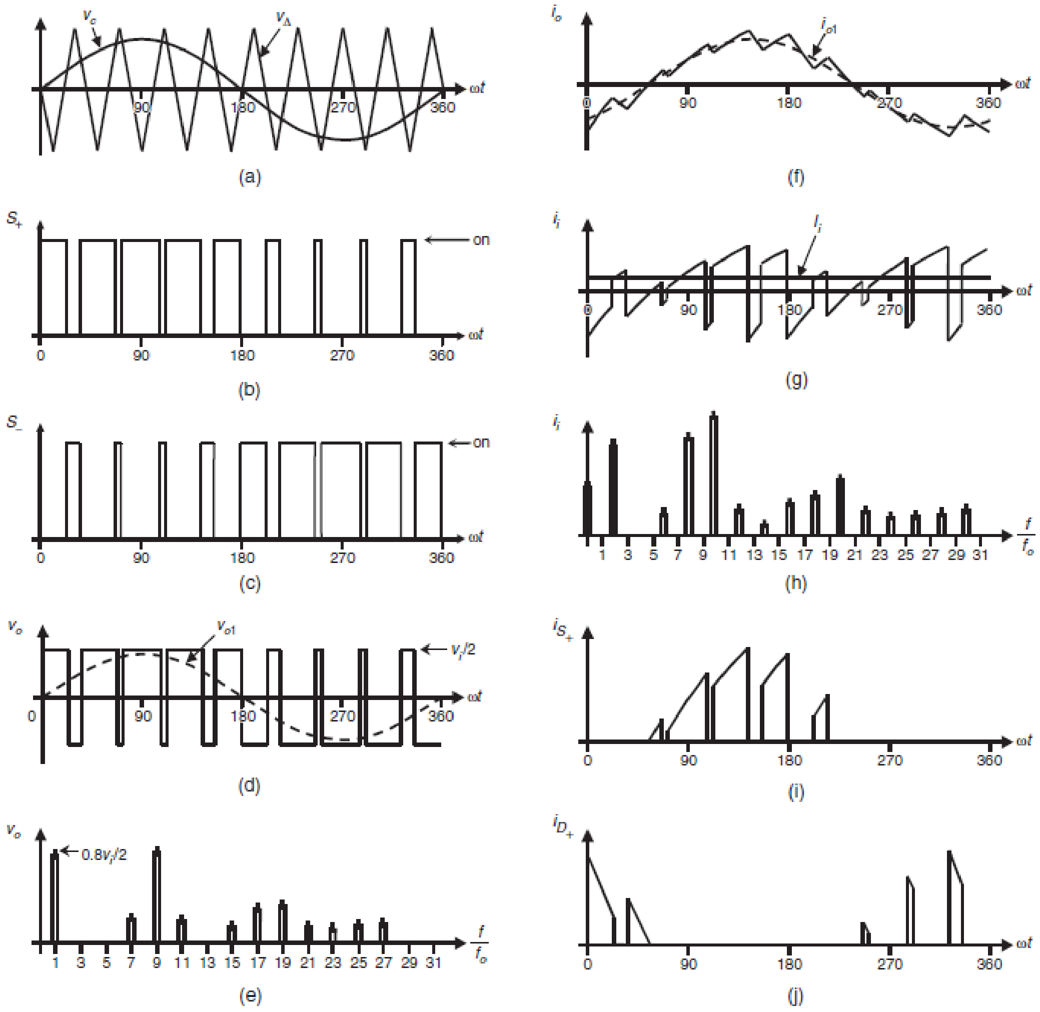


Fig. 21 The half bridge VSI waveform for the SPWM (ma =0.8, mf =9) (a) carrier and modulating signals (b) S+ state (c) S- state (d) Ac output (e) Ac output spectrum (f) Ac output current (g) dc current (h) dc current spectrum (i) switch S+ current (j) D + current

Square-wave Modulating Technique

Both switches S+ and S− are on for one half-cycle of the ac output period. This is equivalent to the SPWM technique with an infinite modulation index m_a. The fundamental ac output voltage features amplitude given by

$$\dot{V}_{o1} = \dot{V}_{aN1} = \frac{4}{\pi}\frac{V_i}{2} \qquad (14)$$

And the harmonics feature an amplitude given by

$$\dot{V}_{oh} = \frac{\dot{V}_{o1}}{h} \qquad (15)$$

Fig. 22 shows output voltage of the half bridge VSI, for square wave modulation technique.

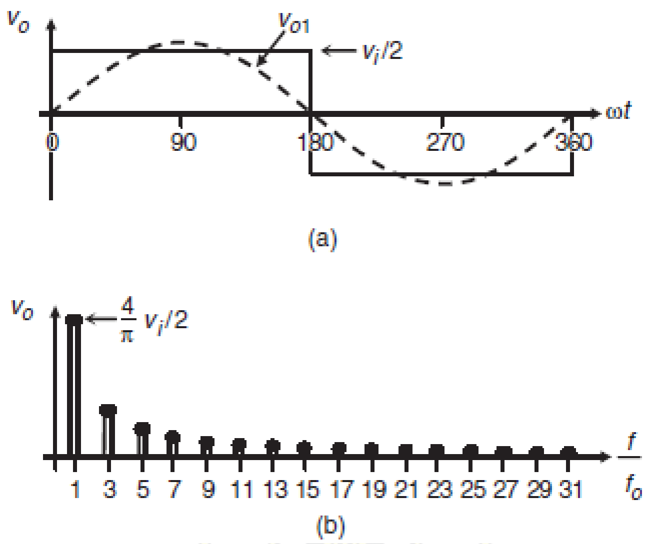


Fig. 22 The half bridge VSI, for square wave modulation technique

Full-bridge VSI

Figure 23 shows the power topology of a full-bridge VSI. This inverter is similar to the half-bridge inverter; however, a second leg provides the neutral point to the load. As expected, both switches S1+ and S1− (or S2+ and S2−) cannot be on simultaneously because a short circuit across the dc link voltage source V_i would be produced. There are four defined (states 1, 2, 3, and 4) and one undefined (state 5) switch state as shown in Table 2. The undefined condition should be avoided so as to be always capable of defining the Ac output voltage always. In order to avoid the short circuit across the dc bus and the undefined Ac output voltage condition, the modulating technique should ensure that either the top or the bottom switch of each leg is on at any instant. It can be observed that the Ac output voltage can take values up to the dc link value V_i, which is twice that obtained with half-bridge VSI topologies. Several modulating techniques have been developed that are applicable to full-bridge VSI_s. Among them are the PWM (bipolar and unipolar) techniques.

Bipolar PWM Technique

States 1 and 2 (Table 2) are used to generate the Ac output voltage in this approach. Thus, the Ac output voltage waveform features only two values, which are V_i and $-V_i$. To generate the states, a carrier-based technique can be used as in half-bridge configurations (Fig. 20), where only one sinusoidal modulating signal has been used. It should be noted that the on-state in switch S+ in the half-bridge corresponds to both switches S_1+ and S_2− being in the on-state in the full-bridge configuration. Similarly, S− in the on-state in the half-bridge corresponds to both switches S_1− and S_2+ being in the on-state in the full-bridge configuration. This is called bipolar carrier-based SPWM.

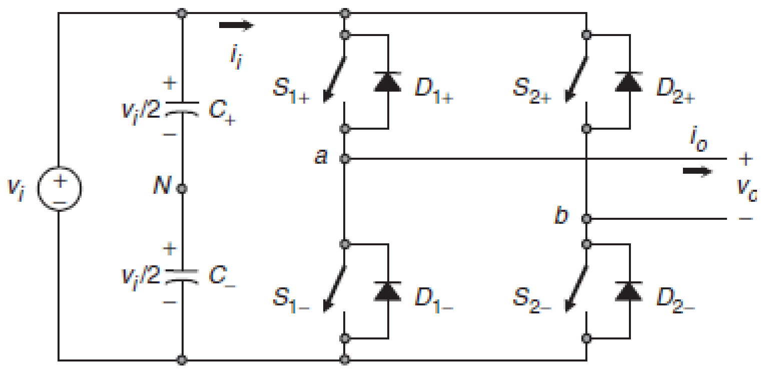


Fig. 23 Single phase full bridge VSI

Table 2 Switch states for a full bridge single phase VSI

State	State #	v_{aN}	v_{bN}	v_o	Components conducting	
S_{1+} and S_{2-} are on and S_{1-} and S_{2+} are off	1	$v_i/2$	$-v_i/2$	v_i	S_{1+} and S_{2-} D_{1+} and D_{2-}	if $i_o > 0$ if $i_o < 0$
S_{1-} and S_{2+} are on and S_{1+} and S_{2-} are off	2	$-v_i/2$	$v_i/2$	$-v_i$	D_{1-} and D_{2+} S_{1-} and S_{2+}	if $i_o > 0$ if $i_o < 0$
S_{1+} and S_{2+} are on and S_{1-} and S_{2-} are off	3	$v_i/2$	$v_i/2$	0	S_{1+} and D_{2+} D_{1+} and S_{2+}	if $i_o > 0$ if $i_o < 0$
S_{1-} and S_{2-} are on and S_{1+} and S_{2+} are off	4	$-v_i/2$	$-v_i/2$	0	D_{1-} and S_{2-} S_{1-} and D_{2-}	if $i_o > 0$ if $i_o < 0$
S_{1-}, S_{2-}, S_{1+}, and S_{2+} are all off	5	$-v_i/2$ $v_i/2$	$v_i/2$ $-v_i/2$	v_i $-v_i$	D_{1-} and D_{2+} D_{1+} and D_{2-}	if $i_o > 0$ if $i_o < 0$

The Ac output voltage waveform in a full-bridge VSI is basically a sinusoidal waveform that features a fundamental component of amplitude ˆvo1 that satisfies the expression

$$\dot{V_{o1}} = \dot{V_{ab1}} = V_i\, m_a \qquad (16)$$

In the linear region of the modulating technique ($m_a \le 1$), which is twice that obtained in the half-bridge VSI. Identical conclusions can be drawn for the frequencies and the amplitudes of the harmonics in the Ac output voltage and dc link current, and for operations at smaller and larger values of odd mf (including the over-modulation region ($m_a > 1$)), than in half-bridge VSIs, but considering that the maximum ac output voltage is the dc link voltage V_i . Thus, in the over-modulation region the fundamental component of amplitude $\dot{V_{o1}}$ satisfies the expression

$$Vi < \dot{V_{o1}} = \dot{V_{ab1}} < \frac{4}{\pi} V_i \qquad (17)$$

Chapter 5

Control System Modeling and Representation

Objectives

- Presentation of the main mathematics required for modeling, control systems.
- Providing methods and tools for proper modeling and representation of physical systems.

Introduction

The Laplace transform is an operational method that can be used for solving linear differential equations. By use of Laplace transforms, we can convert many common functions, such as sinusoidal functions, damped sinusoidal functions, and exponential functions, into algebraic functions of complex variables. Operations such as differentiation and integration can be replaced by algebraic operations in the complex plane. Therefore, a linear differential equation can be transformed into an algebraic equation in a complex variable *s*. If the algebraic equation in *s* is solved for the dependent variable, then the solution of the differential equation (the inverse Laplace transform of the dependent variable) may be found by use of a Laplace transform table or by use of the partial-fraction expansion technique. An advantage of the Laplace transform method is that it allows the use of graphical technique for predicting the system performance without actually solving system differential equations. Another advantage of the Laplace transform method is that, when we solve the differential equation, both the transient component and steady-state components of the solution can be obtained simultaneously. Laplace transform is followed by a discussion of aspects of modeling control system components and an introduction to the concept of transfer functions. Representation of control systems using block diagrams is covered to conclude this chapter.

The Laplace Transform

The *Laplace transform* is one of the most indispensable tools at the disposal of the control systems engineering. The fundamental definition of the Laplace transform *X(s)* of a time-varying function *x(t)* is given by the basic relationship

$$X(s) = \int_0^{\infty} x(t)e^{-st}dt \qquad (1)$$

Note that the operation on the right-hand side of Equation (1) involves the multiplication of the variable function *x(t)*, which is assumed to be defined from *t = 0* to infinity by the factor e^{-st}. The new variable *(s)* is called the *Laplace operator*. The resulting expression is clearly a function of *s* only, as the integration is between zero and infinity. A commonly used notation to express the Laplace transform operation is

$$X(s) = \mathcal{L}\{x(t)\} \qquad (2)$$

The Laplace transform of some common functions are obtained in the following examples. Table 1 lists the Laplace transform of functions encountered in control systems engineering.

TABLE 1
SOME LAPLACE TRANSFORM PAIRS

X(t)	*X(s)*
Unit impulse *δ(t)*	1
Unit step *u(t)*	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
$e^{-\lambda t}$	$\frac{1}{s+\lambda}$
$t^n e^{-\lambda t}$	$\frac{n!}{(s+\lambda)^{n+1}}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$e^{-\lambda t}\sin \omega t$	$\frac{\omega}{(s+\lambda)^2+\omega^2}$
$e^{-\lambda t}\cos \omega t$	$\frac{s+\lambda}{(s+\lambda)^2+\omega^2}$

Example 1

Find the Laplace transform of the function

$$x(t) = e^{\lambda t} \qquad t \geq 0$$

Solution

Using Eq. (1), we can write

$$X(s) = \int_0^\infty e^{\lambda t} e^{-st} dt$$

Thus

$$X(s) = \int_0^\infty e^{-(s-\lambda)t} dt$$

$$= \frac{e^{-(s-\lambda)t}}{s-\lambda} |_\infty^0$$

$$= \frac{1}{s-\lambda}$$

We can thus conclude that

$$\mathcal{L}\{e^{\lambda t}\} = \frac{1}{s-\lambda} \qquad (3)$$

The result of Example 1 is useful in many ways. First, the Laplace transform of the exponential function is important and should be memorized. Second, Eq. (3) can be used to derive the Laplace transform of other functions. To start, note that a step function *u(t)* is defined as

$$u(t) = 1 \qquad t \geq 0$$
$$u(t) = 0 \qquad t \leq 0$$

Note that

$$u(t) = e^{0t} \qquad (4)$$

Thus substituting $\lambda = 0$ into Eq. (3) gives us

$$\mathcal{L}\{u(t)\} = \frac{1}{s} \qquad (5)$$

Linear property of the Laplace transform

Consider two functions $x_1(t)$ and $x_2(t)$, and two scalars a_1 and a_2 . Define the sum x_3 by

$$x_3(t) = a_1x_1(t) + a_2x_2(t)$$

The Laplace transform of $x_3(t)$ is given by

$$X_3(s) = \int_0^\infty [a_1x_1(t) + a_2x_2(t)]\, e^{-st}dt$$

$$X_3(s) = a_1\int_0^\infty x_1(t)e^{-st}dt + a_2\int_0^\infty x_2(t)e^{-st}dt$$

Thus we conclude that

$$X_3(s) = a_1X_1(s) + a_2X_2(s)$$

This proves the linear property of the Laplace transform stated as

$$\mathcal{L}\{a_1x_1(t) + a_2x_2(t)\} = a_1\mathcal{L}\{x_1(t)\} + a_2\mathcal{L}\{x_2(t)\} \qquad (6)$$

We can combine (6) with the result of Example 1 to derive the Laplace transform of other functions.

Example 2

Find the Laplace transform of

$$x(t) = A\sin(\omega t + \alpha)$$

Solution

First recall Euler's identity,

$$e^{j\theta} = \cos\theta + j\sin\theta$$

Thus

$$e^{-j\theta} = \cos\theta - j\sin\theta$$

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

where, $j = \sqrt{-1}$.

We can thus write

$$x(t) = \frac{A}{2j}\left[e^{j(\omega t+\alpha)} - e^{-j(\omega t+\alpha)}\right]$$

$$x(t) = a_1 e^{\lambda_1 t} + a_2 e^{\lambda_2 t}$$

Clearly we have

$$a_1 = \frac{A}{2j}e^{j\alpha}$$

$$a_2 = -\frac{A}{2j}e^{-j\alpha}$$

$$\lambda_1 = j\omega$$

$$\lambda_2 = -j\omega$$

On the basis of Eq. (6)

$$\mathcal{L}\{a_1x_1(t) + a_2x_2(t)\} = a_1\mathcal{L}\{x_1(t)\} + a_2\mathcal{L}\{x_2(t)\} \qquad (6)$$

And Eq. (3)

$$\mathcal{L}\{e^{\lambda t}\} = \frac{1}{s-\lambda} \qquad (3)$$

$$X(s) = \frac{a_1}{s-\lambda_1} + \frac{a_2}{s-\lambda_2}$$

This is reduced to,

$$X(s) = \frac{A}{2j}\left(\frac{e^{j\alpha}}{s-j\omega} - \frac{e^{-j\alpha}}{s+j\omega}\right)$$

$$X(s) = \frac{A}{2j}\frac{s\left(e^{j\alpha} - e^{-j\alpha}\right) + j\omega\left(e^{j\alpha} + e^{-j\alpha}\right)}{s^2+\omega^2}$$

$$X(s) = A\frac{s\sin\propto + \omega\cos\alpha}{s^2+\omega^2}$$

Thus,

$$\mathcal{L}\{A\,sin(\omega t + \alpha)\} = A\frac{s\sin\propto + \omega\cos\alpha}{s^2+\omega^2} \qquad (7)$$

If $a = 0$, we obtain

$$\mathcal{L}\{A \sin \omega t\} = A\,\frac{\omega}{s^2+\omega^2} \quad (8)$$

For $a = \pi/2$, we obtain

$$\mathcal{L}\{A \cos \omega t\} = A\,\frac{s}{s^2+\omega^2} \quad (9)$$

The Inverse Laplace Transform

Consider the simple differential equation

$$a_2\frac{d^2x}{dt^2} + a_1\frac{dx}{dt} + a_0x = u(t)$$

Application of the Laplace transform to both sides, assuming zero initial conditions, gives us

$$(a_2s^2 + a_1s + a_0)X(s) = U(s)$$

This is an algebraic equation which can be written as

$$X(s) = \frac{U(s)}{a_2s^2 + a_1s + a_0}$$

Suppose now that the input function *u(t)* is a unit step; thus

$$U(s) = \frac{1}{s}$$

As a result, the Laplace transform of *x(t)* is given by

$$X(s) = \frac{1}{s(a_2s^2 + a_1s + a_0)}$$

Finding the function *x(t)* whose transform is as given above, is symbolized by the inverse transform operator $\mathcal{L}^{-1}$; Thus

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s(a_2s^2 + a_1s + a_0)}\right\}$$

A formal definition of the *inverse Laplace transform* is given by

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = \frac{1}{2\pi j}\int_{\alpha-j\infty}^{\alpha+j\infty} X(s)e^{st}\,ds \quad (10)$$

Where α is a real constant.

It is quite possible (although somewhat difficult) to obtain the inverse Laplace transform by performing the integration indicated in Eq. (11). A much more effective way is commonly employed in control systems engineering, which relies on performing a partial fraction expansion of *X(s)* as sum of the functions $X_1(s)$, $X_2(s)$, … , $X_n(s)$.

$$X(s) = X_1(s) + X_2(s) + \cdots + X_n(s) \quad (11)$$

The functions $X_1(s)$, $X_2(s)$, … , $X_n(s)$ can then be looked up in a table of Laplace transform pairs and hence we can obtain the corresponding inverse $x_1(t)$, $x_2(t)$, … , $x_n(t)$. The final result is then

$$x(t) = x_1(t) + x_2(t) + \cdots + x_n(t) \tag{12}$$

Partial Fraction Expansion Method

In Control systems analysis, F(s), the Laplace transform of *f(t)*, frequently occurs in the form

$$F(s) = \frac{B(s)}{A(s)}$$

where, *A(s)* and *B(s)* are polynomials in *s*. In the expansion of *F(s)* = *B(s)*/*A(s)* into a partial-fraction form, it is important that the highest power of *s* in *A(s)* be greater than the highest power of *s* in *B(s)*. If such is not the case, the numerator *B(s)* must be divided by the denominator *A(s)* in order to produce a polynomial in *s* plus a remainder. (This remainder is a ratio of polynomials in *s* whose numerator is of lower degree than the denominator.) If *F(s)* is broken up into components,

$$F(s) = F_1(s) + F_2(s) + \ \ldots + F_n(s)$$

and if the inverse Laplace transforms of $F_1(s)$, $F_2(s)$, …, $F_n(s)$ are readily available, then

$$\mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}[F_1(s)] + \mathcal{L}^{-1}[F_2(s)] + \cdots + \mathcal{L}^{-1}[F_n(s)]$$

$$\mathcal{L}^{-1}[F(s)] = f_1(t) + f_2(t) + \cdots + f_n(t) \tag{13}$$

where, $f_1(t)$, $f_2(t)$, …, $f_n(t)$ are the inverse Laplace transform of $F_1(s)$, $F_2(s)$, …, $F_n(s)$, respectively. The inverse Laplace transform of *F(x)* thus obtained is unique except possibly at points where the time function is discontinuous. Whenever the time function *f(t)* and its Laplace transform *F(s)* have a one-to-one correspondence. The advantage of the partial-fraction expansion approach is that the individual terms of *F(s)*, resulting from the expansion into partial fraction form, are very simple functions of *s*; consequently, it is not necessary to refer to a Laplace transform table if we memorize several simple Laplace transform pairs. It should be noted, however, that in applying the partial fraction expansion technique in the search for the inverse Laplace transform of *F(s)* = *B(s)*/*A(s)* the roots of the denominator polynomial *A(s)* must be obtained in advance. That is, this method does not apply until the denominator polynomial has been factored.

Partial Fraction Expansion when F(s) Involves Distinct Poles Only.

Consider *F(s)* written in the factored form

$$F(s) = \frac{B(s)}{A(s)} = \frac{K(s+z_1)(s+z_2)\dots(s+z_m)}{(s+p_1)(s+p_2)\dots(s+p_n)} \qquad \text{for } m < n$$

where, p_1, p_2, …,p_n and z_1, z_2, …, z_m are either real or complex quantities, but for each complex p_i or z_j there will occur the complex conjugate of p_i or z_j, respectively.

If *F(s)* involves distinct poles only, then it can be expanded into a sum of simple partial fractions as follows:

$$F(s) = \frac{B(s)}{A(s)} = \frac{a_1}{s+p_1} + \frac{a_2}{s+p_2} + \dots + \frac{a_n}{s+p_n} \tag{14}$$

where, a_k ($k = 1, 2, \dots, n$) are constants.

The coefficient a_k is called the residue at the pole $s = -p_k$. The value of a_k can be found by multiplying both sides of equation (2-14) by $(s + p_k)$ and letting $s = -p_k$, which gives

$$\left[(s+p_k)\frac{B(s)}{A(s)}\right]_{s=-p_k} = \left[\frac{a_1}{s+p_1}(s+p_k) + \frac{a_2}{s+p_2}(s+p_k) + \dots + \frac{a_k}{s+p_k}(s+p_k) + \dots + \frac{a_n}{s+p_n}(s+p_k)\right]_{s=-p_k} = a_k$$

We see that all the expanded terms drop out with the exception of a_k. Thus the residue a_k is found from

$$a_k = \left[(s+p_k)\frac{B(s)}{A(s)}\right]_{s=-p_k} \tag{15}$$

Note that, since *f(t)* is a real function of time, if p_1 and p_2 are complex conjugates, then the residues a_1 and a_2 are also complex conjugates. Only one of the conjugates a_1 or a_2 needs to be evaluated because the other is known automatically.

- Referring to Eq. (14) and noting that

$$\mathcal{L}^{-1}\left[\frac{a_k}{s+p_k}\right] = a_k e^{-p_k t}$$

f(t) obtained as

$$f(t) = \mathcal{L}^{-1}[F(s)] = a_1 e^{-p_1 t} + a_1 e^{-p_1 t} + \dots + a_n e^{-p_n t} \qquad \text{for } t \geq 0$$

Example 3

Find the inverse Laplace transform of

$$F(s) = \frac{s+3}{(s+1)(s+2)}$$

Solution

The partial fraction expansion of *F(s)* is

$$F(s) = \frac{s+3}{(s+1)(s+2)} = \frac{a_1}{s+1} + \frac{a_2}{s+2}$$

where, a_1 and a_2 are found by using equation (2-26):

$$a_1 = \left[(s+1)\frac{s+3}{(s+1)(s+2)}\right]_{s=-1} = \left[\frac{s+3}{s+2}\right]_{s=-1} = 2$$

$$a_2 = \left[(s+2)\frac{s+3}{(s+1)(s+2)}\right]_{s=-2} = \left[\frac{s+3}{s+1}\right]_{s=-2} = -1$$

Thus

$$f(t) = \mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}\left[\frac{2}{s+1}\right] + \mathcal{L}^{-1}\left[\frac{-1}{s+2}\right]$$

$$f(t) = 2e^{-1t} - e^{-2t} \qquad \text{for } t \geq 0$$

Example 4

Find the inverse Laplace Transform of

$$F(s) = \frac{1}{(s+2)(s+5)(s+11)}$$

Solution

We write

$$F(s) = \frac{A_1}{s+2} + \frac{A_2}{s+5} + \frac{A_3}{s+11}$$

Thus

$$A_1(s+5)(s+11) + A_2(s+2)(s+11) + A_3(s+2)(s+5) = 1$$

Put

$s = -2;$ $\qquad A_1 = 1/27$

Put

$s = -5;$ $\qquad A_2 = -1/18$

Put

$s = -11;$ $\qquad\qquad$ $A_3 = 1/54$

As a result,

$$F(s) = \frac{1}{54}\left(\frac{2}{s+2} - \frac{3}{s+5} + \frac{1}{s+11}\right)$$

The inverse Laplace transform is thus

$$f(t) = \frac{1}{54}\left(2e^{-2t} - 3e^{-5t} + 1e^{-11t}\right)$$

Example 5

Obtain the inverse Laplace transform of

$$G(s) = \frac{s^3 + 5s^2 + 9s + 7}{(s+1)(s+2)}$$

Solution

Here, since the degree of the numerator polynomial is higher than that of the denominator polynomial, we must divide the numerator by the denominator

$$\begin{array}{r|l}
 & s \ + 2 \\ \hline
s^2 + 3s + 2 & s^3 + 5s^2 + 9s + 7 \\
 & s^3 + 3s^2 + 2s \\ \hline
 & 0 \ + 2s^2 + 7s + 7 \\
 & \quad 2s^2 + 6s + 4 \\ \hline
 & \quad 0 \ + \ s \ + 3
\end{array}$$

$$G(s) = s + 2 + \frac{s+3}{(s+1)(s+2)}$$

Note that the Laplace transform of the unit impulse function $\delta(t)$ is 1, and that the Laplace transform of $d\delta/dt$ is s.The third term on the right hand side of the last equation is $F(s)$ in Example 2-3. So the inverse Laplace transform of G(s) is given as

$$g(t) = \frac{d}{dt}\delta(t) + 2\delta(t) + 2e^{-1t} - e^{-2t} \qquad \text{for } t \geq 0$$

Example 6

Find the inverse Laplace transform of

$$F(s) = \frac{2s+12}{s^2 + 2s + 5}$$

Solution

Notice that the denominator polynomial can be factored as

$$s^2 + 2s + 5 = (s + 1 + j2)(s + 1 - j2)$$

If the function *F(s)* involves a pair of complex-conjugate poles, it is convenient not to expand *F(s)* into partial fractions but to expand it into the sum of a *damped sine* and a *damped cosine* function. Noting that $s^2 + 2s + 5 = (s + 1)^2 + 2^2$ and referring to the Laplace transform of $e^{-at}\sin\omega t$ and $e^{-at}\cos\omega t$, rewritten thus,

$$\mathcal{L}\{e^{-\alpha t}\sin\omega t\} = \frac{\omega}{(s+\alpha)^2 + \omega^2}$$

$$\mathcal{L}\{e^{-\alpha t}\cos\omega t\} = \frac{s+\alpha}{(s+\alpha)^2 + \omega^2}$$

The given *F(s)* can be written as a sum of a *damped sine* and a *damped cosine* functions

$$F(s) = \frac{2s + 12}{s^2 + 2s + 5} = \frac{10 + 2(s+1)}{(s+1)^2 + 2^2}$$

$$F(s) = 5\frac{2}{(s+1)^2 + 2^2} + 2\frac{s+1}{(s+1)^2 + 2^2}$$

It follows that

$$f(t) = \mathcal{L}^{-1}\{F(s)\}$$

$$f(t) = 5\mathcal{L}^{-1}\left[\frac{2}{(s+1)^2 + 2^2}\right] + 2\mathcal{L}^{-1}\left[\frac{s+1}{(s+1)^2 + 2^2}\right]$$

$$f(t) = 5e^{-t}\sin 2t + 2e^{-t}cos2t$$

Partial Fraction Expansion when F(s) Involves Multiple Poles.

Instead of discussing the general case, we shall use an example to show how to obtain the partial fraction expansion of *F(s)*. Consider the following *F(s)*.

$$F(s) = \frac{s^2 + 2s + 3}{(s+1)^3}$$

The partial fraction expansion of this F(s) involves three terms

$$F(s) = \frac{B(s)}{A(s)} = \frac{b_1}{s+1} + \frac{b_2}{(s+1)^2} + \frac{b_3}{(s+1)^3}$$

where, b_3, b_2, and b_1 are determined as follows.

By multiplying both sides of the last equation by $(s+1)^3$, we have

$$(s+1)^3 \frac{B(s)}{A(s)} = b_1(s+1)^2 + b_2(s+1) + b_3 \qquad (16)$$

Then letting $s = -1$, Eq. (2-16) gives

$$\left[(s+1)^3 \frac{B(s)}{A(s)}\right]_{s=-1} = b_3$$

Also differentiation of both sides of Eq. (2-16) with respect to s yields.

$$\frac{d}{ds}\left[(s+1)^3 \frac{B(s)}{A(s)}\right] = b_2 + 2b_1(s+1) \qquad (17)$$

- If we let s=-1 in Eq. (2-17), then

$$\frac{d}{ds}\left[(s+1)^3 \frac{B(s)}{A(s)}\right]_{s=-1} = b_2$$

- By differentiating both sides of Eq. (2-17) with respect to s, the result is

$$\frac{d^2}{dx^2}\left[(s+1)^3 \frac{B(s)}{A(s)}\right] = 2b_1$$

- From the preceding analysis it can be seen that the values of b_3, b_2 and b_1 are found systematically as follows:

$$b_3 = \left[(s+1)^3 \frac{B(s)}{A(s)}\right]_{s=-1} = (s^2+2s+3)_{s=-1} = 2$$

$$b_2 = \left\{\frac{d}{ds}\left[(s+1)^3 \frac{B(s)}{A(s)}\right]\right\}_{s=-1} = \left[\frac{d}{ds}(s^2+2s+3)\right]_{s=-1}$$

$$b_2 = (2s+2)_{s=-1} = 0$$

$$b_1 = \frac{1}{2!}\left\{\frac{d^2}{ds^2}\left[(s+1)^3 \frac{B(s)}{A(s)}\right]\right\}_{s=-1} = \frac{1}{2!}\left[\frac{d^2}{ds^2}(s^2+2s+3)\right]_{s=-1} = \frac{1}{2}(2) = 1$$

We thus obtain

$$f(t) = \mathcal{L}^{-1}[F(s)]$$

$$f(t) = \mathcal{L}^{-1}\left[\frac{1}{s+1}\right] + \mathcal{L}^{-1}\left[\frac{0}{(s+1)^2}\right] + \mathcal{L}^{-1}\left[\frac{2}{(s+1)^3}\right]$$

$$f(t) = e^{-1t} + 0 + t^2e^{-1t}$$

$$f(t) = (1+t^2)e^{-1t} \qquad \text{for } t \geq 0$$

Example 7

Find the Laplace transform of the following function

$$F(s) = \frac{1}{s^2(s+3)}$$

Solution

We write

$$F(s) = \frac{A_1}{s} + \frac{A_2}{s^2} + \frac{A_3}{s+3}$$

Thus

$$A_1s(s+3) + A_2(s+3) + A_3s^2 = 1$$

Put

$s = 0;$ $\qquad A_2 = 1/3$

Put

$s = -3;$ $\qquad A_3 = 1/9$

To obtain A_1 we have to equate coefficient of s^2 on both sides to obtain

$$A_1 + A_3 = 0$$

Thus

$$A_1 = -1/9$$

As a result

$$F(s) = \frac{1}{9}\left(-\frac{1}{s} + \frac{3}{s^2} + \frac{1}{s+3}\right)$$

The inverse Laplace transform is thus given by

$$f(t) = \frac{1}{9}(-1 + 3t + e^{-3t})$$

Example 8

Find the Laplace transform of the following function

$$F(s) = \frac{1}{s(s+3)^2}$$

Solution

We can write

$$F(s)=\frac{A_1}{s}+\frac{A_2}{s+3}+\frac{A_3}{(s+3)^2}$$

Thus

$$A_1(s+3)^2+A_2s(s+3)+A_3s=1$$

Put

$s = 0;$ $\qquad A_1 = 1/9$

Put

$s = -3;$ $\qquad A_3 = -1/3$

The coefficient of s^2 on both sides yields

$$A_1+A_2=0$$

Thus

$$A_2 = -1/9$$

As a result,

$$F(s)=\frac{1}{9}\left[\frac{1}{s}-\frac{1}{s+3}-\frac{3}{(s+3)^2}\right]$$

The inverse Laplace Transform is thus given by

$$f(t)=\frac{1}{9}(1-e^{-3t}-3te^{-3t})$$

Solution of Linear Time-Invariant, Differential Equations

The Laplace transform can be used for solving linear time-invariant differential equations with known initial conditions. In solving differential equations two steps are involved.

- **Step 1.** By taking the Laplace transform of each term in the given differential equation. Convert the differential equation into an algebraic equation in *s*. Obtain the expression for the Laplace transform of the dependant variable by rearranging the algebraic equation.
- **Step 2.** The time solution of the differential equation is obtained by finding the inverse Laplace transform of the dependant variable. In the following discussion, two examples are used to demonstrate the solution of linear, time-invariant, differential equation by the Laplace transform method.

Example 9

Find the solution x(t) of the differential equation

$$\ddot{x}+3\dot{x}+2x=0 \qquad x(0)=a \qquad \dot{x}(0)=b$$

where, *a* and *b* are constants.

Solution

By writing the Laplace transform of x(t) as X(s) or

$$\mathcal{L}[x(t)] = X(s)$$

we obtain

$$\mathcal{L}[\dot{x}] = sX(s) - x(0)$$

$$\mathcal{L}[\ddot{x}] = s^2X(s) - sx(0) - \dot{x}(0)$$

And so the given differential equation becomes

$$[s^2X(s) - sx(0) - \dot{x}(0)] + 3[sX(s) - x(0)] + 2X(s) = 0$$

By substituting the given initial conditions into this last equation, we obtain

$$[s^2X(s) - as - b] + 3[sX(s) - a] + 2X(s) = 0$$

Or

$$(s^2 + 3s + 2)X(s) = as + b + 3a$$

Solving for *X(s)*, we have

$$X(s) = \frac{as + b + 3a}{s^2 + 3s + 2} = \frac{as + b + 3a}{(s+1)(s+2)} = \frac{2a + b}{s+1} - \frac{a+b}{s+2}$$

The inverse Laplace transform of *X(s)* gives

$$x(t) = \mathcal{L}^{-1}\left[\frac{2a+b}{s+1}\right] - \mathcal{L}^{-1}\left[\frac{a+b}{s+2}\right]$$

$x(t) = (2a+b)e^{-t} - (a+b)e^{-2t},$ for $t \geq 0$

which is the solution of the given differential equation.
Notice that the initial conditions *a* and *b* appear in the solution. Thus *x(t)* has no undetermined constants.

Example 10

Find the solution *x(t)* of the differential equation

$$\ddot{x} + 2\dot{x} + 5x = 3 \qquad x(0) = 0 \qquad \dot{x}(0) = 0$$

Solution

Noting that $\mathcal{L}[3] = 3/s,\ x(0) = 0,$ and $\dot{x}(0) = 0,$
The Laplace transform of the differential equation becomes

$$s^2X(s) + 2sX(s) + 5X(s) = \frac{3}{s}$$

$$X(s) = \frac{3}{s(s^2+2s+5)} = \frac{3}{5}\frac{1}{s} - \frac{3}{5}\frac{s+2}{s^2+2s+5}$$

$$X(s) = \frac{3}{5}\frac{1}{s} - \frac{3}{10}\frac{2}{(s+1)^2+2^2} - \frac{3}{5}\frac{s+1}{(s+1)^2+2^2}$$

Hence the inverse Laplace transform becomes

$$x(t) = \mathcal{L}^{-1}[X(s)]$$

$$x(t) = \frac{3}{5}\mathcal{L}^{-1}\left[\frac{1}{s}\right] - \frac{3}{10}\mathcal{L}^{-1}\left[\frac{2}{(s+1)^2+2^2}\right] - \frac{3}{5}\mathcal{L}^{-1}\left[\frac{s+1}{(s+1)^2+2^2}\right]$$

$x(t) = \frac{3}{5} - \frac{3}{10}e^{-t}sin2t - \frac{3}{5}e^{-t}\cos 2t$ for $t \geq 0$

which is the solution of the given differential equation.

Elementary Physical System Models

Control system engineers use physical laws that describe the interaction between variables of interest in the system under consideration. We use the simple laws of the system elements. In this lecture we discuss several laws of simple elements.

Passive Elements

- Passive elements in electric circuits are the resistance R, the inductance L, and the capacitance C. The relation between the voltage across $v(t)$ and current through $i(t)$ depends on the element.
- For a resistance element we have

$$v(t) = Ri(t)$$

Taking Laplace transform:

$$V(s) = RI(s)$$

- For an inductance we have

$$v(t) = L\frac{di(t)}{dt}$$

Taking the Laplace transform with zero initial condition,

$$V(s) = sLI(s)$$

- For a capacitance we have

$$v(s) = \frac{1}{C}\int i(t)\,dt$$

Taking the Laplace transform with zero initial condition,

$$V(s) = \frac{1}{Cs}I(s)$$

Kirchhoff's laws are also useful (described in a later chapter).

Transfer Functions

A transfer function is defined as the ratio of the Laplace transform of the output variable in a linear time-invariant dynamic system to the Laplace transform of the input variable with zero initial conditions. A number of examples is given below, together with the derivations associated with simple dynamic systems.

- ***Direct Current Generator***

Figure 1 shows a diagram of separately excited dc generator. The input voltage $v_i(t)$ produces a current $i_f(t)$ in the field circuit.

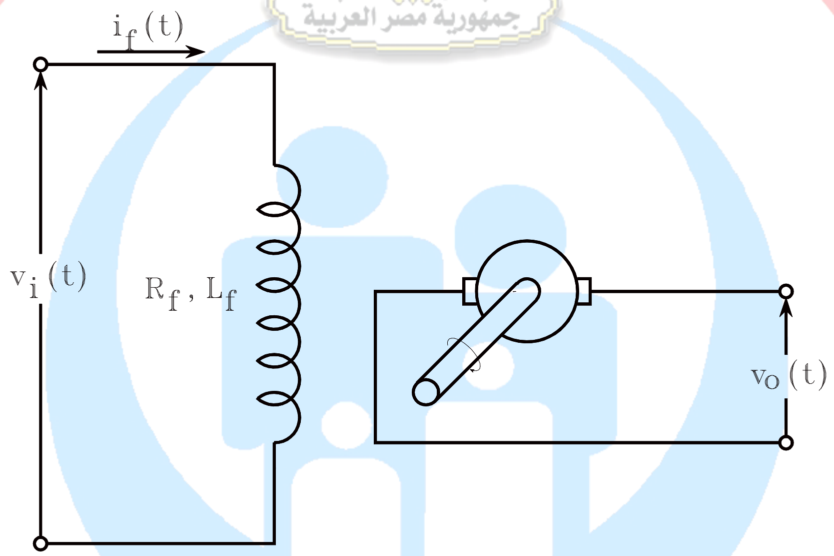


Figure 1: Schematic diagram of a dc generator.

Assuming that the field circuit resistance is R_f and the inductance is L_f, we can write

$$v_i(t) = R_f i_f(t) + L_f \frac{di_f(t)}{dt}$$

Assuming that the magnetization characteristic of the machine is linear in the region of interest, and that the generator is driven at constant speed, we can write the output voltage as

$$v_o(t) = K_g i_f(t)$$

Where, K_g is proportionality constant.

Now taking the Laplace Transform of the foregoing two equations we find that:

$$V_i(s) = (R_f + L_f s) I_f(s)$$
$$V_o(s) = K_g I_f(s)$$

As a result, the transfer function between input and output voltages is given by

$$\frac{V_o(s)}{V_t(s)} = \frac{K_g}{R_f + L_f s}$$

- ***Amplidyne***

The amplidyne is a two-stage dc generator, as shown in Figure 2.

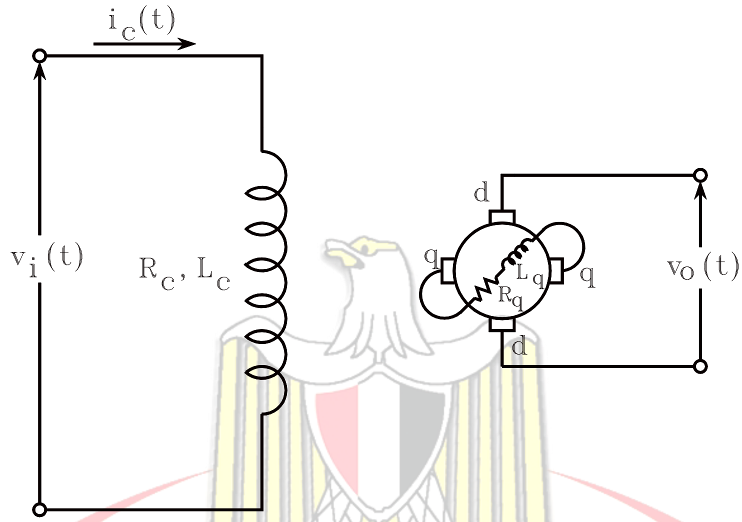


Figure 2. The amplidyne.

Assume that the control winding's resistance and inductance are denoted as R_c and L_c. Let the control (input) voltage is $v_c(t)$ and the current in the control winding is $i_c(t)$. We have

$$v_c(t) = R_c i_c(t) + L_c \frac{di_c}{dt}$$

The electromotive force (emf) developed in the quadrature axis winding is proportional to the field current.
Thus

$$e_q(t) = K_{cq} i_c(t)$$

where K_{cq} is a constant for constant speed of rotation.
The quadrature-axis voltage $e_q(t)$ produces a current $i_q(t)$ assuming that R_q and L_q are the resistance and inductance values for the quadrature-axis winding.
Thus we write

$$e_q(t) = R_q i_q(t) + L_q \frac{di_q}{dt}$$

The quadrature-axis current sets up a flux, which in turn produces the output voltage $v_d(t)$ according to

$$v_d(t) = K_{dq} i_q(t)$$

The Laplace domain equation for the above time domain equations are:

$$I_c(s) = \frac{V_c(s)}{R_c + L_c s}$$

$$E_q(s) = K_{cq} I_c(s)$$

$$I_q(s) = \frac{E_q(s)}{R_q + L_q s}$$

$$V_d(s) = K_{dq} I_q(s)$$

As a result, we have

$$\frac{V_d(s)}{V_c(s)} = \frac{K_{dq} K_{cq}}{(R_q + L_q s)(R_c + L_c s)}$$

which is the transfer function of the amplidyne.

- ***Field Controlled DC Motor***

In a field controlled dc motor (Figure 3), the voltage input is fed to the field, which can be modeled as a resistance R_f in series with the inductance L_f. A field current $i_f(t)$ is established in accordance with

$$v_i(t) = R_f i_f(t) + L_f \frac{di_f}{dt}$$

The field current sets up a flux which together with the rotational speed of the motor develops a back emf $e_m(t)$.

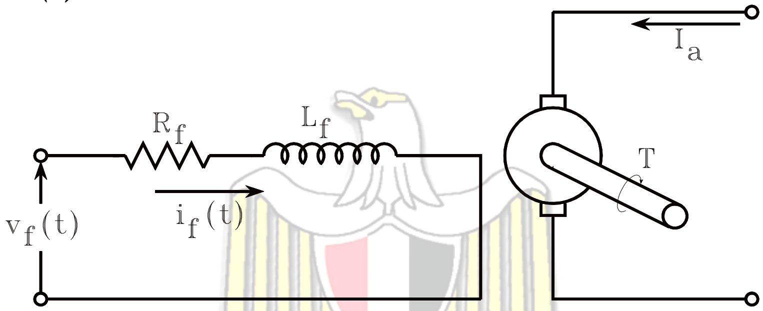


Figure 3. A field controlled dc motor.

Assuming a linear (straight line) relation between e_m and i_f for a given rotational speed ω_m, we can write

$$e_m = K_m i_f(t) \omega_m(t)$$

The parameter K_m is a constant that depends on the motor's design particulars. The motor's developed electrical power is given in terms of the armature current I_a and the back emf as

$$P_e(t) = I_a e_m(t)$$

Note that the armature current is assumed constant. We thus can write

$$P_e(t) = K_m I_a i_f(t) \omega_m(t)$$

Assuming that the electrical power $P_e(t)$ suffers no losses in being transmitted as mechanical power $P_m(t)$, we can write

$$P_e(t) = P_m(t)$$

We know that the mechanical power is the product of torque and angular velocity, thus

$$P_m = T(t) \omega_m(t)$$

- It is thus clear that under the foregoing assumptions, we can write the developed torque as

$$T(t) = K_1 i_f(t)$$

where,

$$K_1 = K_m I_a$$

As a result, we conclude that the motor's torque is proportional to the field current. The torque equation; expressed in the Laplace transform is

$$T(s) = K_1 I_f(s)$$

The relation between the field current and the input voltage is

$$I_f(s) = \frac{V_i(s)}{R_f + L_f s}$$

As a result we conclude that the transfer function between the input voltage and output torque can be written as

$$\frac{T(s)}{V_i(s)} = \frac{K_1}{R_f + L_f s}$$

- ***Armature Controlled DC Motor***

If the input voltage to the dc motor is fed to the armature circuit, while field current is maintained constant, we say that the motor is armature controlled.

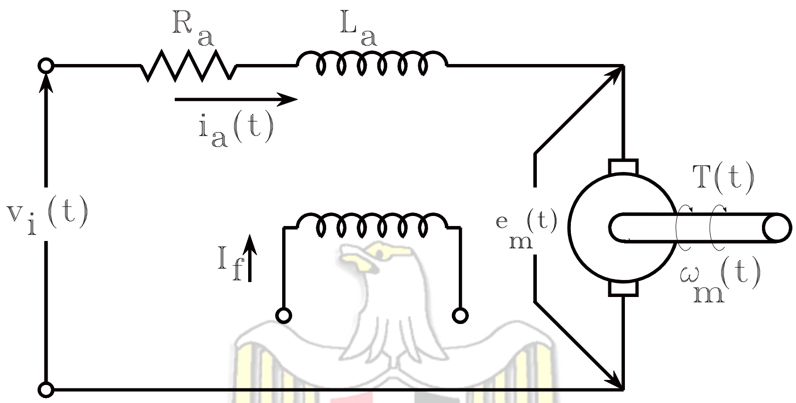


Figure 4. Schematic diagram of an armature-controlled dc motor.

With reference to Figure 4, we can write the loop equation

$$v_i(t) = e_m(t) + R_a i_a(t) + L_a \frac{di_a}{dt}$$

In this equation R_a and L_a are the resistance and the inductance of the armature circuit, respectively.

Thus

$$v_i(t) - e_m(t) = R_a i_a(t) + L_a \frac{di_a}{dt}$$

The armature current is denoted by $i_a(t)$, while the motor's back emf is denoted by $e_m(t)$. The input voltage is denoted by $v_i(t)$. Since the field current is fixed at I_f, we write:

$$e_m(t) = K_m I_f \omega_m(t)$$

The torque developed by the motor under assumptions similar to those stated for the field controlled dc motor can be obtained from

$$P(t) = e_m(t) i_a(t) = T(t)\omega_m(t)$$

Thus

$$T(t) = K_m I_f i_a(t)$$

Let us define an armature controlled motor constant K_a by

$$K_a = K_m I_f$$

As a result, we write

$$e_m(t) = K_a \omega_m(t)$$
$$T(t) = K_a i_a(t)$$

Thus the back emf and torque are proportional to the motor's velocity and armature current, respectively. From the previous discussions we can write the previous equations transformed into Laplace domain.

$$V_i(s) - E_m(s) = (R_a + L_a s) I_a(s)$$

$$I_a(s) = \frac{V_i(s)}{(R_a + L_a s)} - \frac{E_m(s)}{(R_a + L_a s)}$$

$$E_m(s) = K_a \omega_m(s)$$

$$I_a(s) = \frac{V_i(s)}{(R_a + L_a s)} - \frac{K_a \omega_m(s)}{(R_a + L_a s)}$$

$$T(s) = K_a I_a(s)$$

Thus the Laplace transform of the developed torque is given by:

$$T(s) = \frac{K_a V_i(s)}{(R_a + L_a s)} - \frac{K_a^2 \omega_m(s)}{(R_a + L_a s)}$$

This equation is not a single-input output relationship and thus a straightforward transfer function cannot be obtained for *T(s)* in terms of the input voltage. It is possible, however, to obtain a transfer function between $\omega_m(s)$ and $V_i(s)$ as indicated below. Assume that the motor is driving a load such that no opposing torque T_i exists. We can thus write

$$T(s) = (J_{eq} s + B_{eq})\omega_m(s)$$

where, J_{eq} is the moment of inertia of the rotating parts, and B_{eq} is a term denoting friction coefficient.

We can thus eliminate *T(s)* to obtain:

$$(J_{eq} s + B_{eq})\omega_m(s) = \frac{K_a V_i(s)}{(R_a + L_a s)} - \frac{K_a^2 \omega_m(s)}{(R_a + L_a s)}$$

or,

$$\frac{K_a V_i(s)}{(R_a + L_a s)} = \left[(J_{eq} s + B_{eq}) + \frac{K_a^2}{(R_a + L_a s)}\right]\omega_m(s)$$

As a result, a transfer function between $\omega_m(s)$ as output and $V_i(s)$ as input can be written as:

$$\frac{K_a V_i(s)}{(R_a + L_a s)} = \left[\frac{(J_{eq} s + B_{eq})(R_a + L_a s) + K_a^2}{(R_a + L_a s)}\right]\omega_m(s)$$

$$\frac{\omega_m(s)}{V_i(s)} = \frac{K_a}{(R_a + L_a s)}\left[\frac{(R_a + L_a s)}{K_a^2 + (J_{eq} s + B_{eq})(R_a + L_a s)}\right]$$

$$\frac{\omega_m(s)}{V_i(s)} = \frac{K_a}{K_a^2 + (J_{eq} s + B_{eq})(R_a + L_a s)}$$

- ***Two-Phase Servomotor***

In a two-phase servomotor (Figure 5a) the torque decreases linearly with the speed increase for a constant control winding voltage.

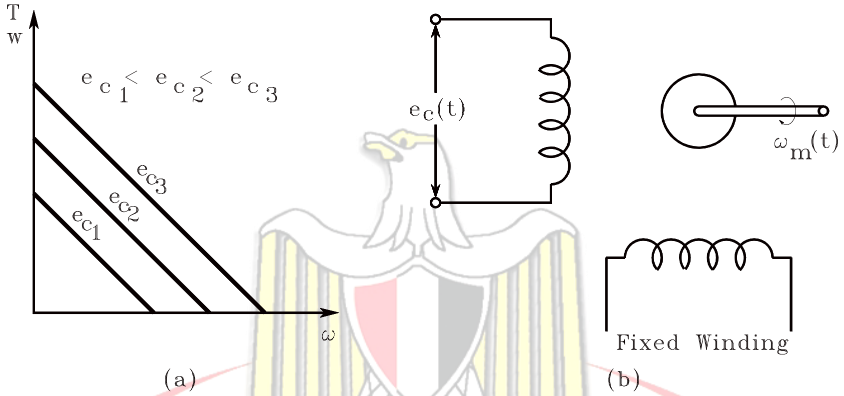


Figure 5. Two-phase servomotor characteristics and diagram

As the control winding voltage is increased, the torque-speed characteristic is raised as shown in Figure 5b. We can thus model the motor's developed torque by

$$T(t) = -K_n\omega(t) + K_c e_c(t)$$

where K_n and K_c are constants.

We know that the speed ω is equal to the time derivative of the angle θ. Thus the last equation can be rewritten as:

$$T(t) = -K_n\dot{\theta}(t) + K_c e_c(t)$$

The torque balance equation for the two-phase servomotor is:

$$T(t) = J\ddot{\theta} + B\dot{\theta}$$

where J is the moment of inertia of the motor and load referred to the motor shaft, and B is the friction coefficient.

Equating the last two relationships:

$$J\ddot{\theta} + (B + K_n)\dot{\theta} = K_c e_c(t)$$

Noting that the control voltage $e_c(t)$ is the input and the displacement angle θ is the output, we see that the transfer function of the system is given by:

$$\frac{\Theta(s)}{E_c(s)} = \frac{K_c}{Js^2 + (B + K_n)s} = \frac{K_c/(B + K_n)}{[Js^2/(B + K_n)] + s}$$

$$\frac{\Theta(s)}{E_c(s)} = \frac{K_c/(B+K_n)}{s\{[Js/(B+K_n)]+1\}} = \frac{K_m}{s(T_m s+1)}$$

where, $K_m = K_c / (B + K_n)$ = motor gain constant, and $T_m = J / (B + K_n)$ = motor time constant

Mechanical Translation Elements

Newton's law as applied to mechanical systems states that the sum of the forces applied to an element is equal to the sum of the reaction forces. There are three basic elements in a mechanical translation system model. These are the mass *M*, the stiffness *K*, and the damping (viscous friction) *B*. A force *F* applied to a mass *M* produces an acceleration *a* of the mass. The reaction force to *F* is given by

$$F = M\,a$$

If the translation is *x*, the acceleration is given by

$$a = \frac{d^2x}{dt^2}$$

Thus for the mass we have

$$F_M = M\frac{d^2x}{dt^2}$$

A diagram showing a mass is given in Figure 6a. The elastance, or stiffness, K of a spring provides a reaction force that is proportional to the deformation Δx of the spring.

$$F_K = K\,\Delta x$$

Note that in Figure 6b,

$$\Delta x = x_1 - x_2$$

Viscous friction or damping B involves energy absorption. The damping force is proportional to the difference of the velocities of the two bodies (Figure 6c).

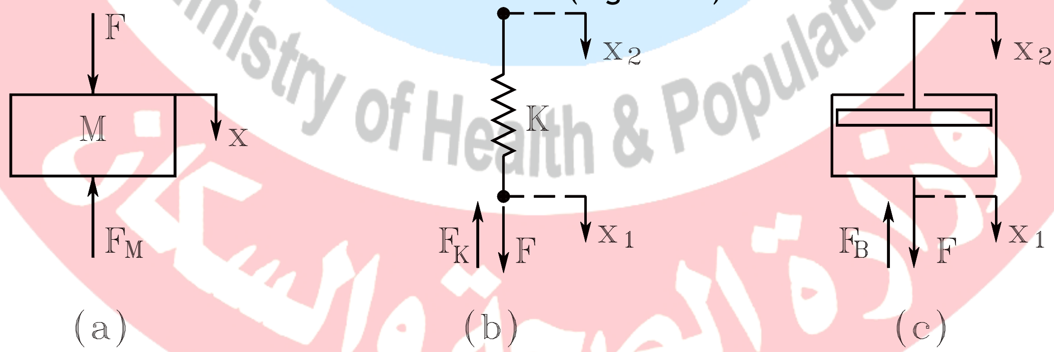


Figure 6. Elements of Mechanical Translation System.

$$F_B = B\left(\frac{dx_1}{dt} - \frac{dx_2}{dt}\right)$$

Mechanical Rotational Elements

Rotational systems involve elements and variables that correspond to those in a translation system (Figure 7). The body's moment of inertia *J* corresponds to the mass *M* in writing the dynamical equation. The angular displacement θ correspond to the translational displacement *x*. The angular velocity ω corresponds to the translation velocity *v*. The angular acceleration corresponds to the translational acceleration a. Torque equations for rotational systems are parallel to force equations in transitional systems. For a body with a moment of inertia J, an applied torque T produces an angular acceleration $d^2\theta/dt^2$. The reaction torque T_J is in opposition to T, with a torque equation given by

$$T_J = J\frac{d^2\theta}{dt^2}$$

The stiffness torque T_K is produced by a spring of stiffness K if it is twisted by angle $\Delta\theta$ by an applied torque T. We have

$$T_K = K(\theta_1 - \theta_2)$$

Damping is encountered wherever a body moves through a fluid. The damping torque T_B is equal to the product of damping B and the relative angular velocity of the damper and is in opposition to the applied torque:

$$T_B = B\left(\frac{d\theta_1}{dt} - \frac{d\theta_2}{dt}\right)$$

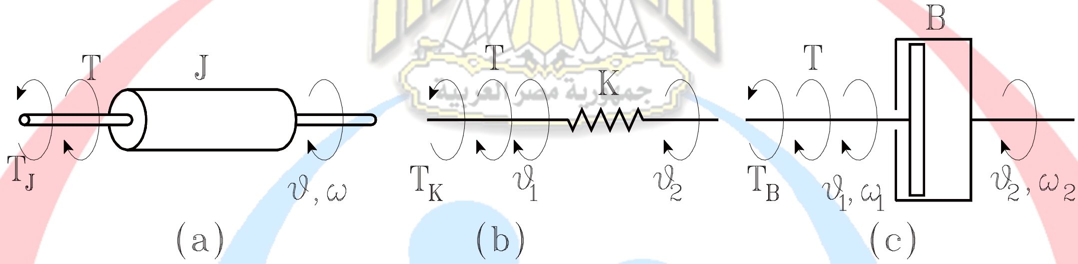


Figure. 7. Elements of Mechanical Rotational Systems

Gear Trains

A normal practice in coupling motors to loads is to employ a gear train to transmit the driving torque to the load. An analysis of such a case is important to evaluate the moment of inertia and damping of the load relative to the motor. Figure 8 shows a driving motor that supplies input torque T_m at an angular velocity ω_m to a load through a gear train having a gear ratio of N_m/N_l. The load torque is denoted by T_l and its angular velocity is denoted by ω_l.

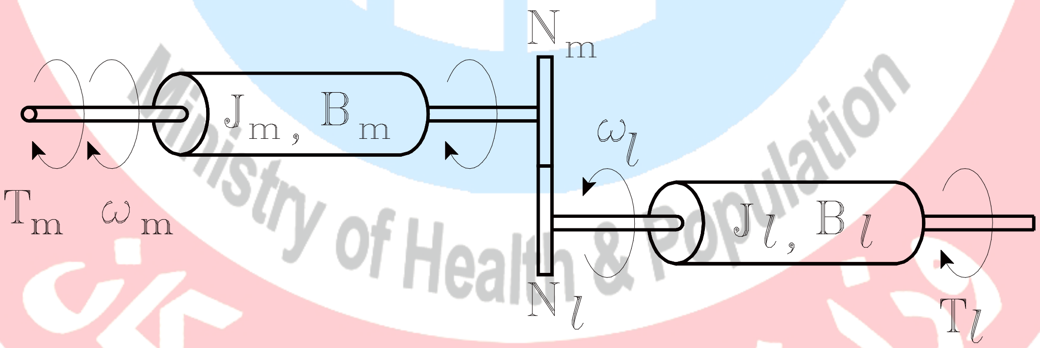


Figure 8. Schematic diagram of a gear train.

The inertia and viscous friction of the motor are denoted by J_m and B_m. In a similar way J_l and B_l are moment of inertia and viscous friction of the load. The torque balance equation on the motor side is given by:

$$T_m = J_m\ddot{\theta}_m + B_m\dot{\theta}_m + T_{ml}$$

Thus the torque provided by the motor is equal to the sum of motor inertial and friction torques and the torque T_{ml} transmitted through the gears. Assuming no power loss in the gears, we can write the power equation:

$$P_{ml} = T_{ml}\omega_m = \tilde{T}_{ml}\omega_l$$

This is a restatement of the fact that power is the product of torque and angular velocity. We also know that:

$$\frac{\omega_m}{\omega_l} = \frac{N_l}{N_m}$$

Thus we can write the driving torque on the load side as

$$\tilde{T}_{ml} = T_{ml}\frac{\omega_m}{\omega_l} = T_{ml}\frac{N_l}{N_m}$$

Now the load side a driving torque balance equation can be written as

$$\tilde{T}_{ml} = J_l\ddot{\theta}_l + B_l\dot{\theta}_l + T_l$$

In terms of the motor's angular velocity $\dot{\theta}_m$ and angular acceleration we thus have

$$\tilde{T}_{ml} = (J_l\ddot{\theta}_m + B_l\dot{\theta}_m)\frac{N_m}{N_l} + T_l$$

Or

$$T_{ml} = \tilde{T}_{ml}\frac{N_m}{N_l} = (J_l\ddot{\theta}_m + B_l\dot{\theta}_m)\left(\frac{N_m}{N_l}\right)^2 + T_l\frac{N_m}{N_l}$$

As a result, we can assert the motor's driving torque is given by

$$T_m = \left(J_m + \left(\frac{N_m}{N_l}\right)^2 J_l\right)\ddot{\theta}_m + \left(B_m + \left(\frac{N_m}{N_l}\right)^2 B_l\right)\dot{\theta}_m + T_l\frac{N_m}{N_l}$$

This result shows that an equivalent moment of inertia J_{eq} and an equivalent viscous friction B_{eq} are experienced by the motor:

$$J_{eq} = J_m + \left(\frac{N_m}{N_l}\right)^2 J_l$$

$$B_{eq} = B_m + \left(\frac{N_m}{N_l}\right)^2 B_l$$

The load torque is seen by the motor as

$$\tilde{T}_l = T_l \frac{N_m}{N_l}$$

Thus the motor's developed torque is given by

$$T_m = J_{eq}\ddot{\theta}_m + B_{eq}\dot{\theta}_m + \tilde{T}_l$$

Some Transfer Functions of physical Systems

A transfer function has been stated as the ration of the Laplace transform of the output variable in a linear time invariant dynamic system to the Laplace transform of the input with zero initial condition. Here is a number of applications.

RC Circuit

- *Consider the RC integrating circuit of Figure 9.*

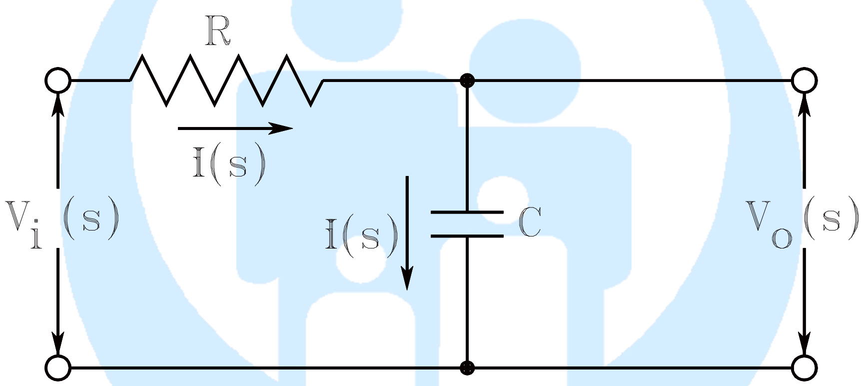


Figure 9. RC integrating circuit

The input voltage is:

$$V_i(s) = \left(R + \frac{1}{Cs}\right) I(s)$$

The output voltage is

$$V_o(s) = \frac{1}{Cs} I(s)$$

The transfer function is thus

$$\frac{V_o(s)}{V_i(s)} = \frac{1}{1 + RCs}$$

➢ *Consider the **RC Differentiating** circuit of Figure 10.*

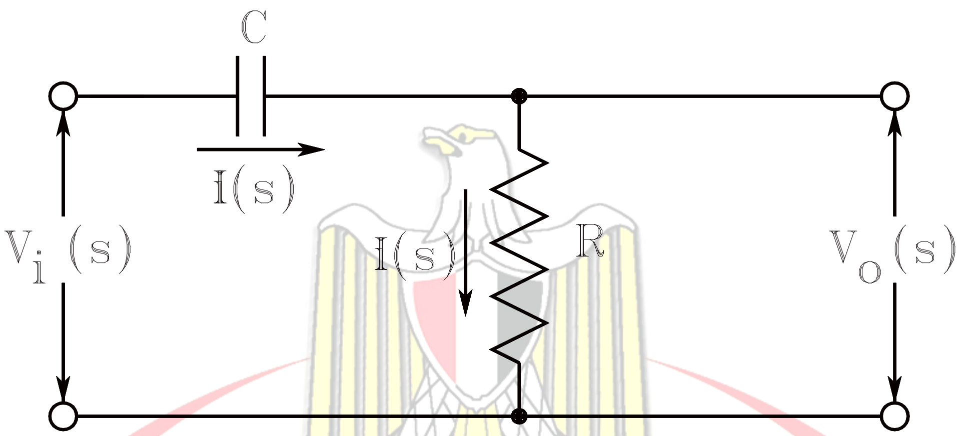


Figure 10. RC differentiating circuit.

$$V_i(s) = \left(R + \frac{1}{Cs}\right) I(s)$$

$$V_o(s) = RI(s)$$

$$\frac{V_o(s)}{V_i(s)} = \frac{1}{R + 1/Cs} = \frac{RCs}{1 + RCs}$$

***Spring-Dashpot** system of Figure 11.*

We can write the force balance equation as

$$B\dot{x}_o + Kx_0 = Kx_i$$

Applying Laplace transform, we have

$$(Bs + K)X_o(s) = KX_i(s)$$

As a result, the transfer function is given by

$$\frac{X_0(s)}{X_i(s)} = \frac{K}{Bs + K}$$

$$\frac{X_0(s)}{X_i(s)} = \frac{1}{1 + (B/K)s}$$

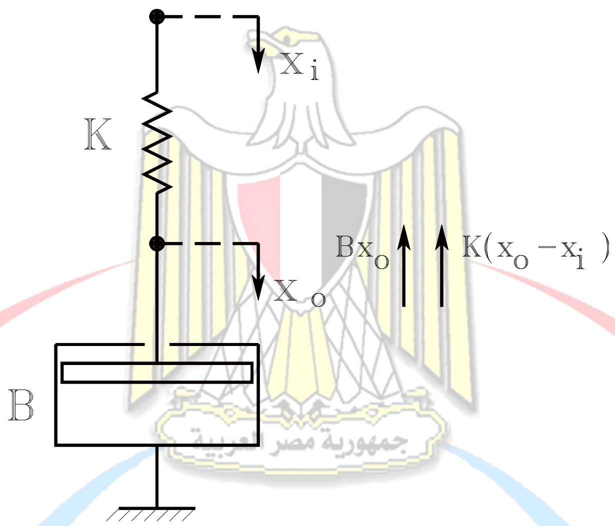


Figure. 11. Spring-Dashpot System.

Note the similarity of this transfer function and that of the RC integrating circuit. Consider the **Spring-Dashpot** system of Figure 11, which has B and K interchanged, as shown in Figure 12. The force balance equation is given by:

$$B(\dot{x}_o - \dot{x}_i) + Kx_o = 0$$

Thus the transfer function can be written as:

$$\frac{X_o(s)}{X_i(s)} = \frac{Bs}{Bs + K}$$

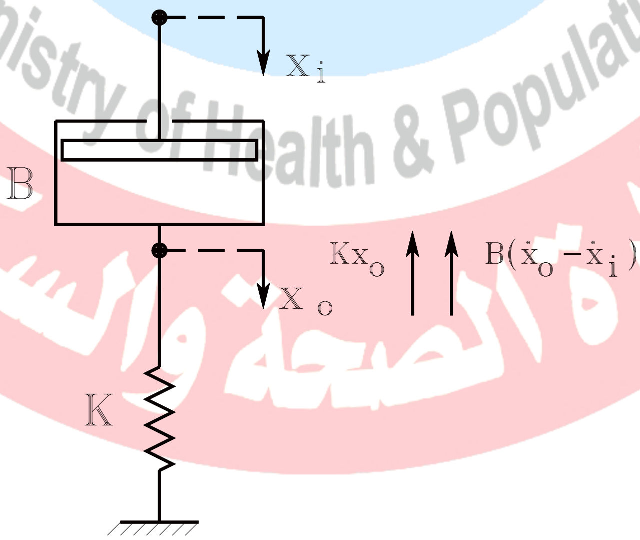


Figure 12. Dashpot-Spring System

The above function is similar to that of the differentiating RC circuit.

***Mechanical rotational** system of Figure 13.*

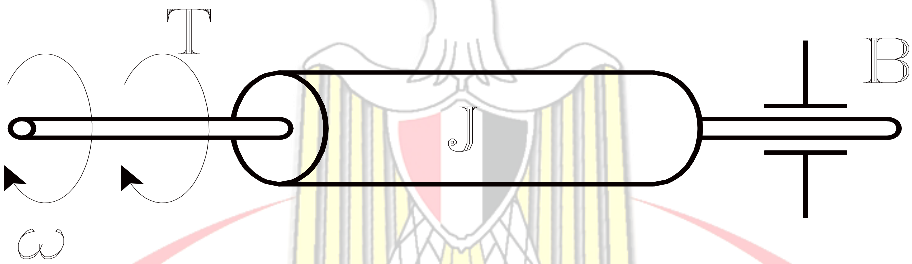


Figure 13. Rotating body

We can write:

$$J\frac{d\omega}{dt}+B\omega=T$$

Thus

$$\frac{\Omega(s)}{T(s)}=\frac{1}{Js+B}$$

The transfer function obtained, relates the transferred output velocity to the transform of the input torque, in the presence of the moment of inertia *J* and viscous friction *B*.

***Mechanical translation** system of Figure 14.*

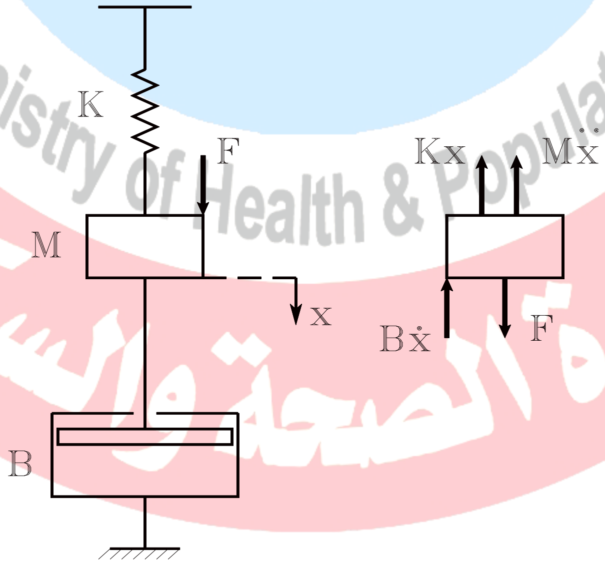


Figure 14. Spring-mass-dashpot system.

The system consists of a spring-mass-dashpot combination. We can write the force balance equation as

$$M\ddot{x} + B\dot{x} + Kx = F$$

As a result, the transfer function is obtained as

$$\frac{X(s)}{F(s)} = \frac{1}{Ms^2 + Bs + K}$$

RLC circuit *of Figure 15.*

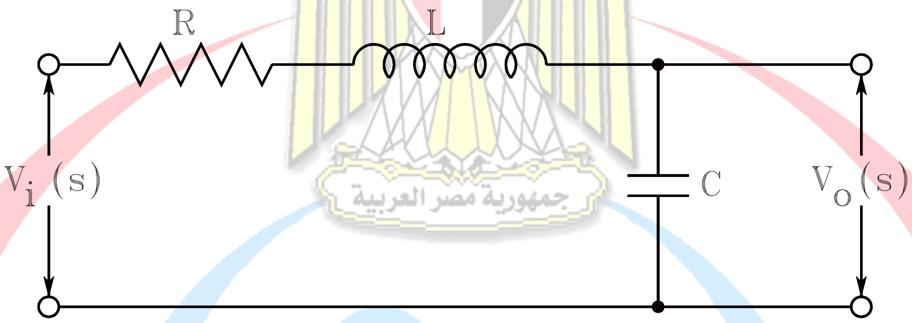


Figure 15. RLC circuit.

$$V_i(s) = \left(R + Ls + \frac{1}{Cs}\right) I(s)$$

$$V_o(s) = \frac{1}{Cs} I(s)$$

Thus the transfer function is given by

$$\frac{V_o(s)}{V_i(s)} = \frac{1}{LCs^2 + RCs + 1}$$

This is very similar to the spring-mass-dashpot system.

Lead-lag RC circuit *of Figure 16.*

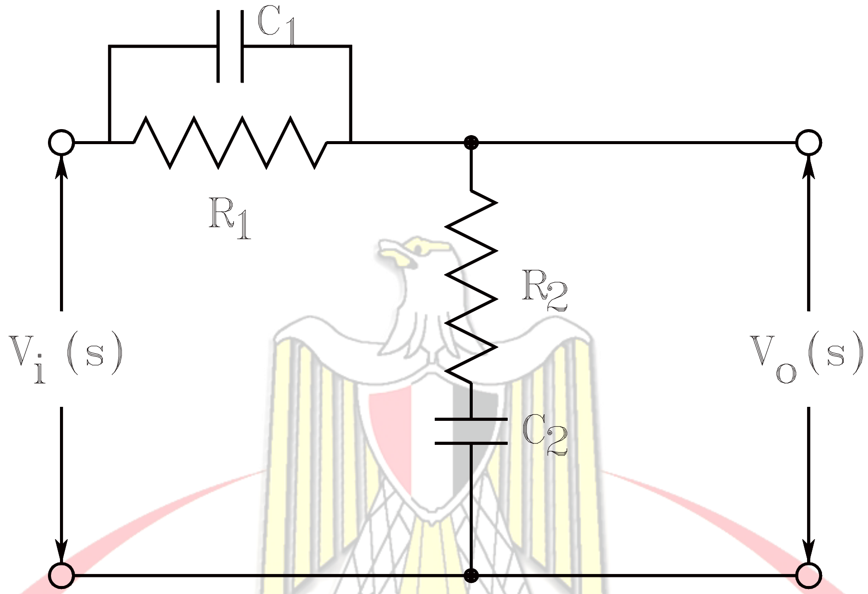


Figure 16. Lead-lag RC circuit

We can write the following impedance function

$$Z_1 = \frac{R_1/C_1 s}{R_1 + 1/Cs} = \frac{R_1}{1 + R_1 C_1 s}$$

Let $\tau_a = R_1 C_1$

Thus

$$Z_1 = \frac{R_1}{1 + \tau_a s}$$

Also

$$Z_2 = R_2 + \frac{1}{C_2 s} = \frac{1 + R_2 C_2 s}{C_2 s}$$

Let

$$\tau_b = R_2 C_2$$

Thus

$$Z_2 = \frac{1 + \tau_b s}{C_2 s}$$

The transfer function is thus given by

$$\frac{V_o(s)}{V_i(s)} = \frac{Z_2}{Z_1 + Z_2} = \frac{(1+\tau_a s)(1+\tau_b s)}{(1+\tau_a s)(1+\tau_b s) + R_1 C_2 s}$$

Let $\tau_{ab} = R_1 C_2;$

thus

$$\frac{V_o(s)}{V_i(s)} = \frac{(1+\tau_a s)(1+\tau_b s)}{\tau_a \tau_b s^2 + (\tau_a + \tau_b + \tau_{ab})s + 1}$$

We can rewrite this equation as

$$\frac{V_o(s)}{V_i(s)} = \frac{1+\tau_a s}{1+\tau_1 s}\frac{1+\tau_b s}{1+\tau_2 s}$$

Where

$$\tau_1 \tau_2 = \tau_a \tau_b$$

$$\tau_1 + \tau_2 = \tau_a + \tau_b + \tau_{ab}$$

Problems

1) Find the Laplace Transform of f(t) defined by:

a) $$\begin{aligned} f(t) &= 0 && for\ t\langle 0 \\ &= te^{-3t}, && for\ \ t \geq 0 \end{aligned}$$

b) $$\begin{aligned} f(t) &= 0 && for\ t\langle 0 \\ &= t^2 e^{-at}, && for\ \ t \geq 0 \end{aligned}$$

c) $$\begin{aligned} f(t) &= 0 && for\ t\langle 0 \\ &= 3\sin(5t + 45°), && for\ \ t \geq 0 \end{aligned}$$

2) Find the inverse Laplace Transform of F(s), where

$$F(s)=\frac{1}{s(s^2+2s+2)}$$

3) Obtain the inverse Laplace transform of

$$F(s)=\frac{5(s+2)}{s^2(s+1)(s+3)}$$

4) Find the inverse Laplace Transform of

$$F(s)=\frac{s^4+2s^3+3s^2+4s+5}{s(s+1)}$$

5) Solve the differential equation:

$$x^{\cdot\cdot}+2x^{\cdot}+10x=t^2,\quad x(0)=0,\quad x^{\cdot}(0)=0$$

Chapter 6

Block Diagrams and Analysis of the Responses

Objectives

- Basics of block diagrams.
- Representation of transfer functions.
- Transfer functions presentation of some practical systems.

Definition

A pictorial representation of the relationships between system variables is offered by the block diagram. In a block diagram three ingredients are commonly present.

1. Functional block

This is a symbol representing the transfer between the input *U(s)* to an element and the output *X(s)* of the element. The block contains the transfer function *G(s)*, as shown in Figure 1.

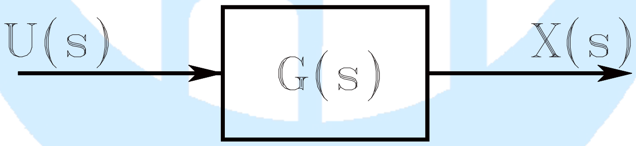


Figure 1. Functional block

The arrow directed into the block represents the input *U(s)*, while that directed out of the block represents the output *X(s)*. The block shown in the figure represents the algebraic relationship

$$X(s) = G(s)\,U(s)$$

2. Summing point

This is a symbol denoted by a circle, the output of which is the algebraic sum of the signals entering into it. A minus sign close to an input signal arrow denotes that this signal is reversed sign in the output expression. Figure 2 shows the relationship

$$E(s) = R(s) - C(s)$$

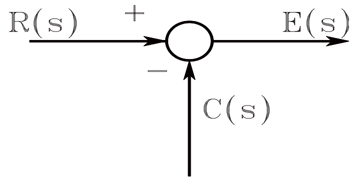


Figure 2. Summing point.

3. Branching (or takeoff) point.

A branching point in a block diagram signifies that the same variable is being utilized elsewhere, as shown in Figure 3.

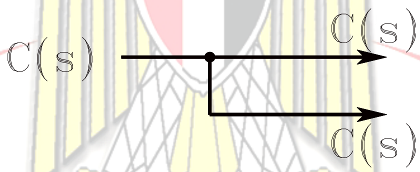


Figure 3. Branching point.

A fundamental block diagram configuration is the single-loop feedback system shown in Figure 4a. The output variable *C(s)* is modified by the feedback element with transfer function *H(s)* to produce the signal *B(s)*:

$$B(s) = C(s)\,H(s)$$

The signal *B(s)* is compared to a reference signal *R(s)* to produce the error signal *E(s)*.

$$E(s) = R(s) - B(s)$$

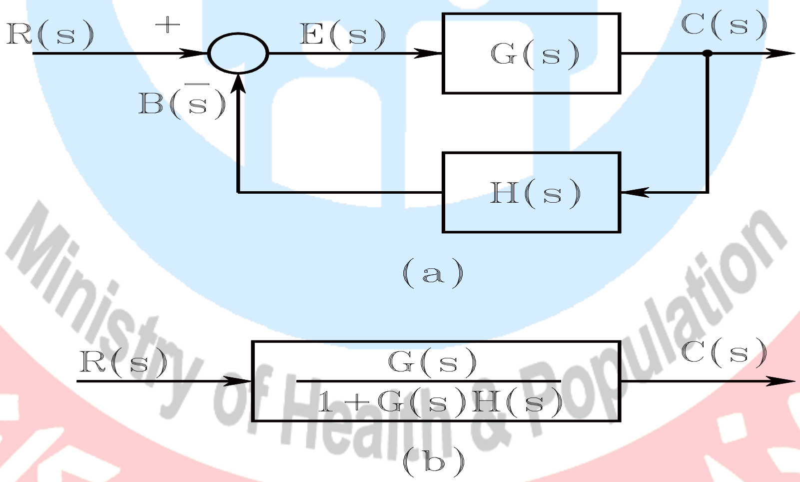


Figure 4. Feedback system. (a) Single-loop; (b) Equivalent of single-loop

The error signal actuates the plant with transfer function *G(s)* to produce the output *C(s)*:

$$C(s) = G(s)\,E(s)$$

Combining the above three equations:

$$E(s) = R(s) - C(s)\,H(s)$$

$$(C(s)/G(s)) = R(s) - C(s)\,H(s)$$

$$R(s) = (C(s)/G(s)) + C(s)\,H(s)$$

$$R(s) = C(s)\left(\frac{1 + G(s)H(s)}{G(s)}\right)$$

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

As a result, we conclude that Figure 4b represents an equivalent of Figure 4a.

To obtain the overall relationship between the outputs and inputs of complex systems, we often have to eliminate variables in the system representation. We consider here the transfer functions of cascaded elements as shown in Figure 5a. We write

$$X_2 = G_1 X_1$$

$$X_3 = G_2 X_2 = G_1 G_2 X_1$$

Thus we can obtain the reduction shown in Figure 5b.
We can further write

$$X_4 = G_3 X_3 = G_1 G_2 G_3 X_1$$

Thus a single equivalent as shown in Figure 3.5c is obtained with

$$G = G_1 G_2 G_3$$

Table 1 shows some important equivalents in block diagram manipulations.

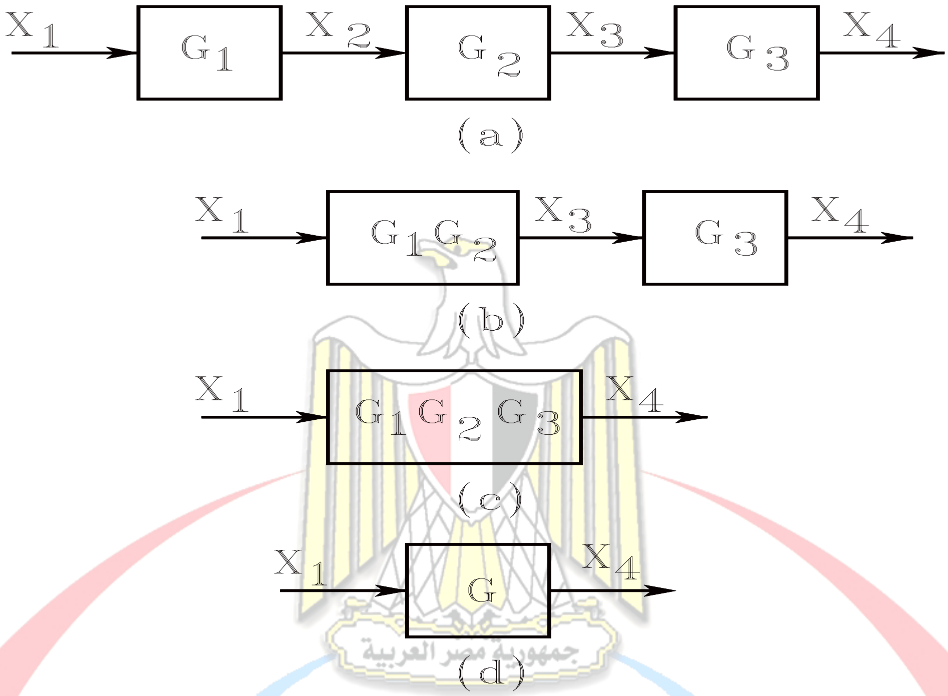
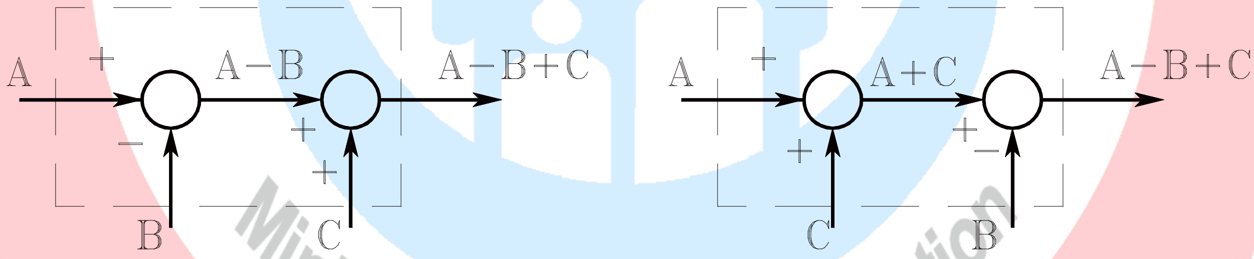


Figure 5. Cascaded system and the steps in its reduction.

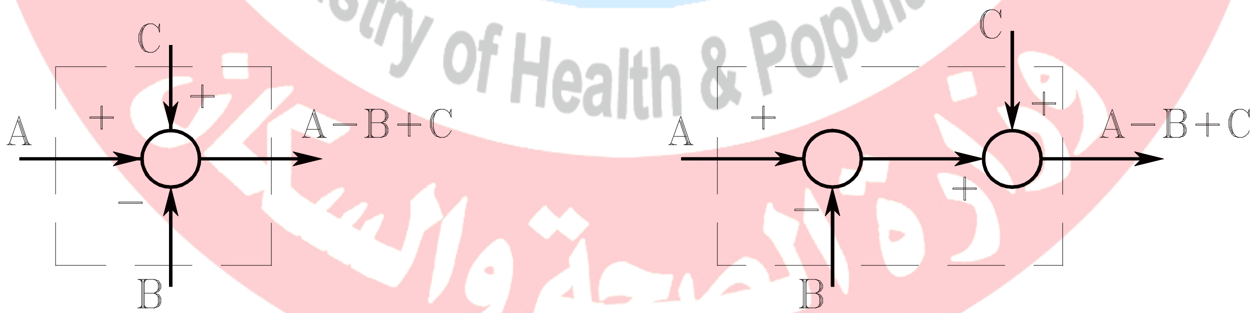
Table 1.

Block diagram reduction equivalents

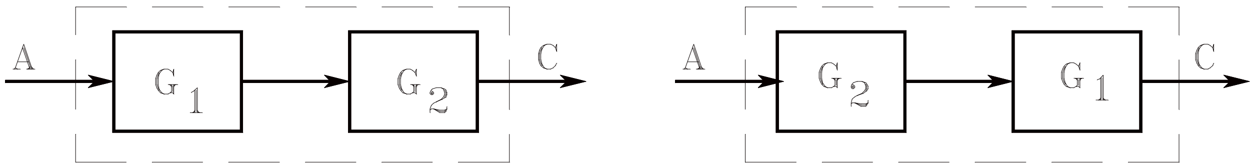
Original Configuration   Alternate Configuration



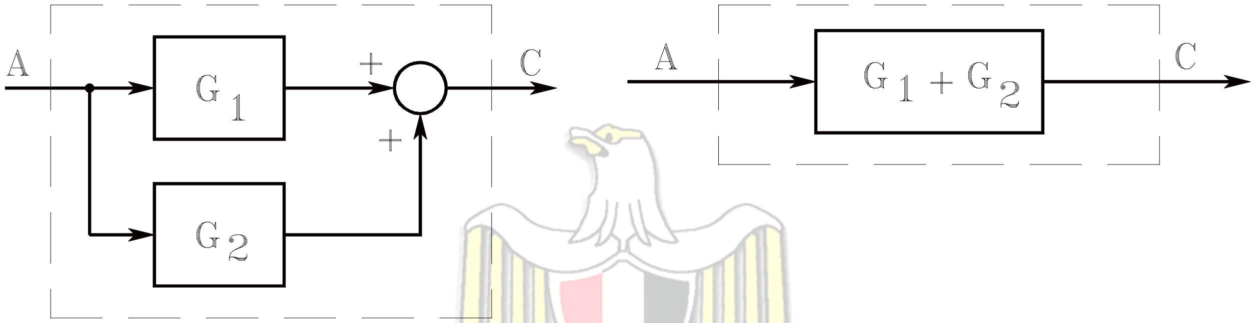
(a)



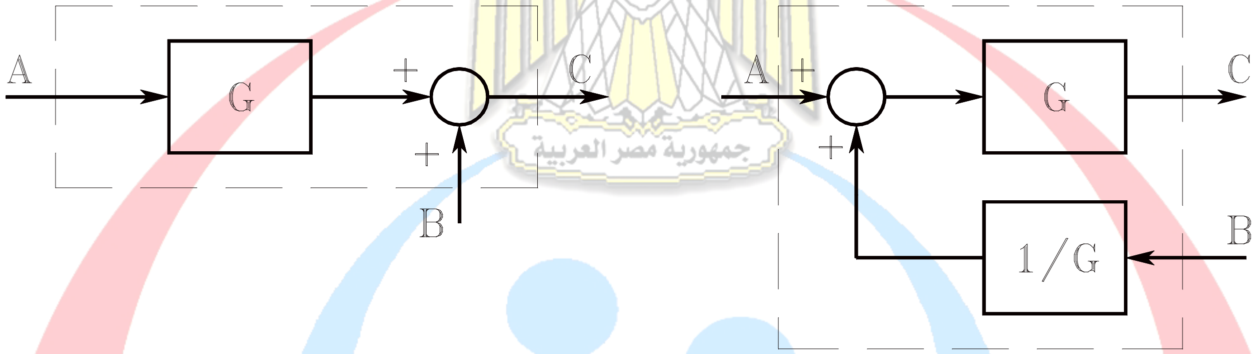
(b)



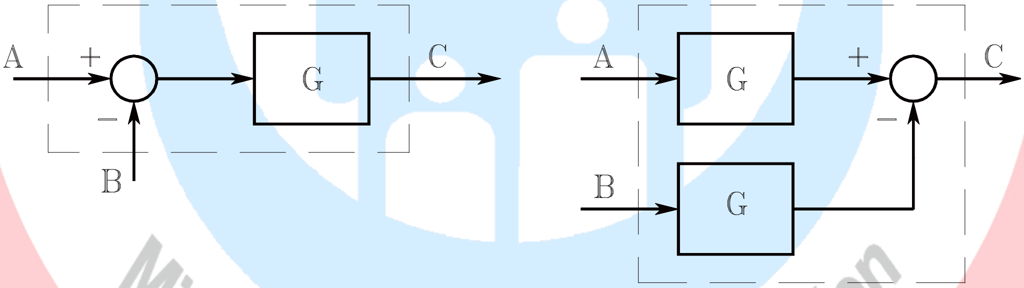
(c)



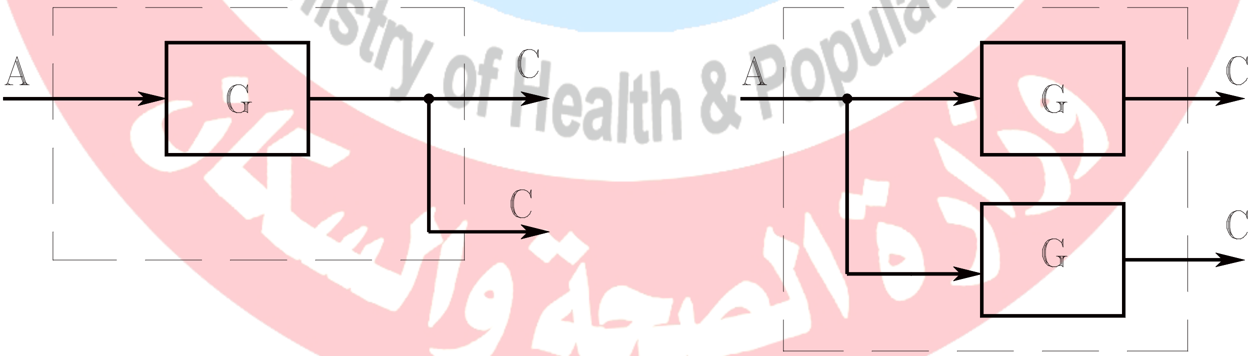
(d)



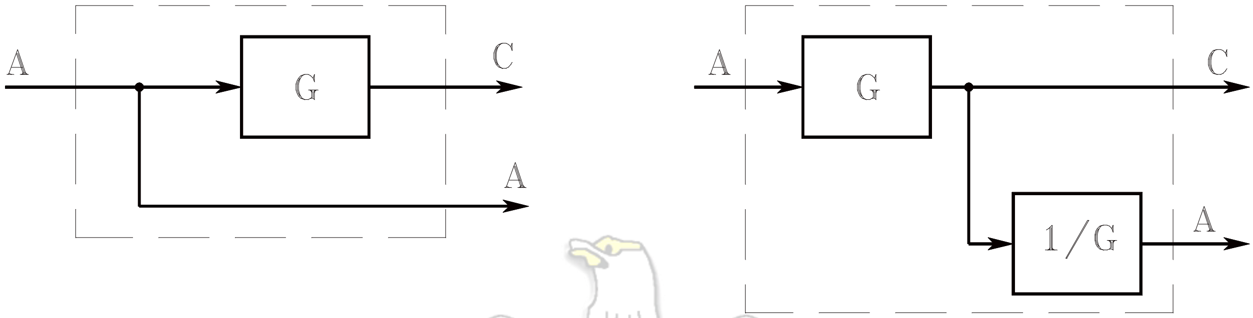
(e)



(f)



(g)



(h)

Example 1.

Simplify the block diagram of Figure 5.6.

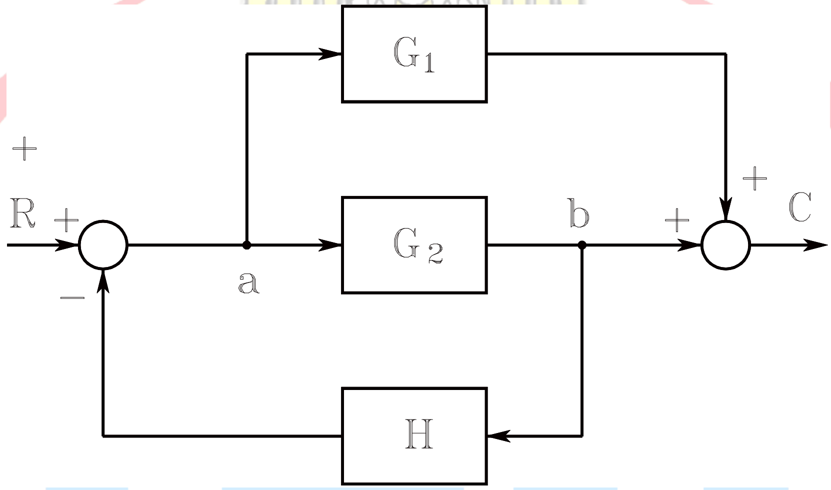


Figure 6. Block diagram or example 1.

Solution

- First, move the branching point *b* to *a* as shown in Figure 7(a).

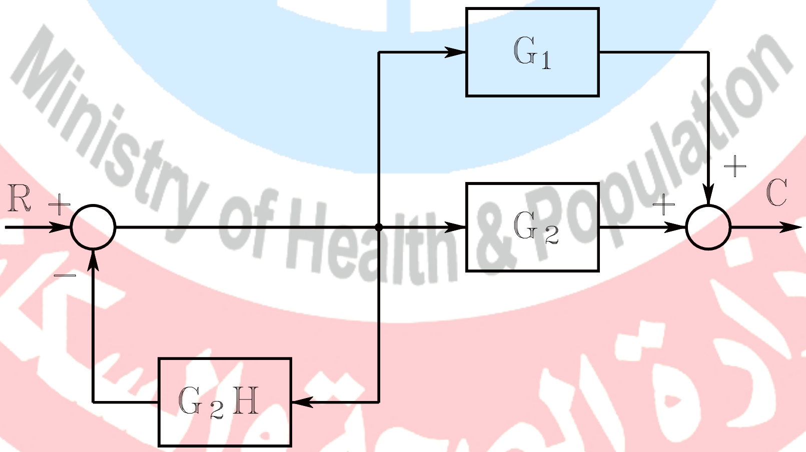


Figure 7(a)

- Now we see that G_1 and G_2 are in parallel, and we reduce the figure to that of Figure 7(b).

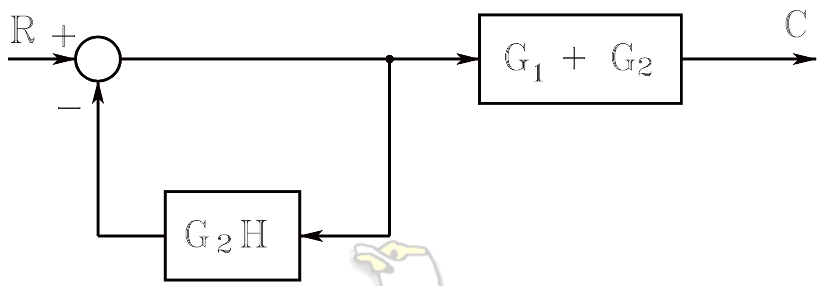


Figure 7(b)

- The feedback loop with a forward gain of 1 and feedback element G_2H can be reduced as shown in Figure 7(c).

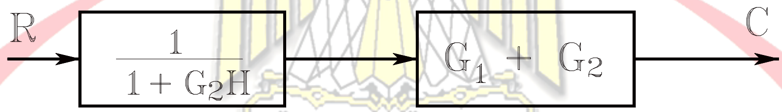


Figure 7(c).

- Finally, Figure 7(d) shows the overall transfer between R and C.

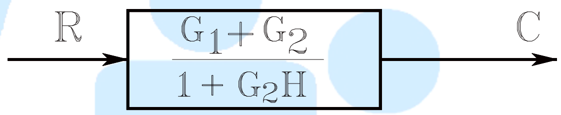


Figure 7(d).

Example 2

Use block diagram reduction techniques to obtain the ratio C/R for the system shown in the block diagram of Figure 8.

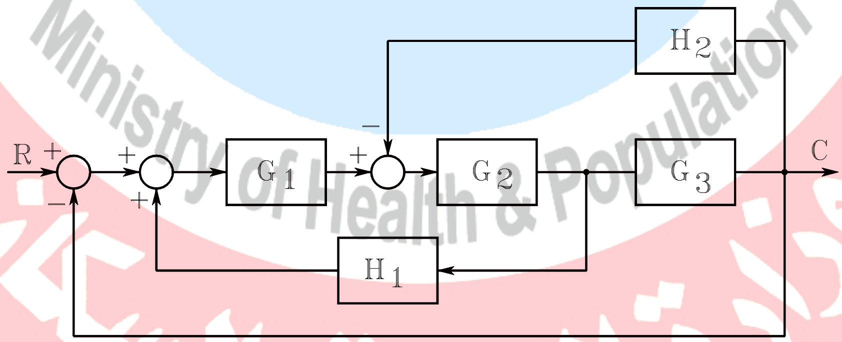


Figure 8. Example 5.2.

Solution

- The following Figure shows the steps of block diagram reduction.

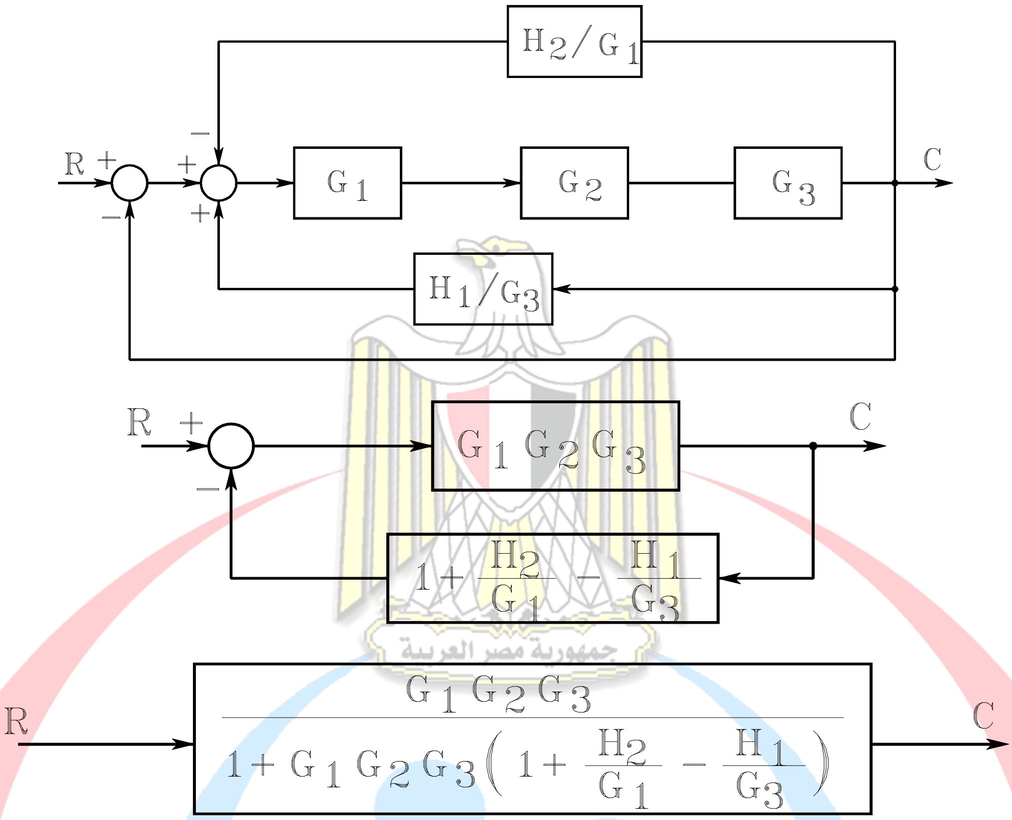


Figure 9. Steps in the foregoing Figure 5.8.

- As a result of this procedure, we conclude that

$$\frac{C}{R} = \frac{G_1 G_2 G_3}{1 + G_1 G_2 G_3 \left(1 + \frac{H_2}{G_1} - \frac{H_1}{G_3}\right)}$$

Block Diagram of a Field-Controlled DC Motor

We have seen that the transfer function between the field voltage and the output torque of a field-controlled DC motor is given by:

$$\frac{T(s)}{V_i(s)} = \frac{K_1}{R_f + L_f s}$$

This is represented in block diagram form in the first block of Figure 10.

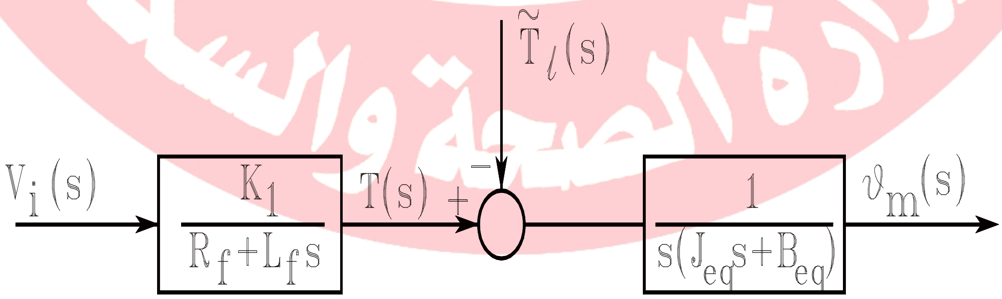


Figure 10. Block diagram of a field-controlled dc motor.

Assume now that the motor is driving a load torque T_l through a gear train of speed ration N_m/N_l . The motor's inertia and viscous friction are J_m and B_m , and the load's inertia and viscous friction are J_l and B_l. We know that

$$T = J_{eq}\ddot{\theta} + B_{eq}\dot{\theta} + \tilde{T}_l$$

In the *s* domain this is written as

$$T(s) = (J_{eq}s^2 + B_{eq}s)\theta_m(s) + \tilde{T}_l(s)$$

This equation is realized in block diagram form by the right-hand block if Figure 5.10.

Block Diagram of Armature-Controlled DC Motor

We know that the developed torque is given by

$$T(s) = \frac{K_a V_i(s)}{R_a + L_a s} - \frac{K_a^2 \omega_m(s)}{R_a + L_a s}$$

With a load torque we write

$$T(s) = (J_{eq}s^2 + B_{eq}s)\theta_m(s) + \tilde{T}_l(s)$$

$$T(s) = (J_{eq}s + B_{eq})\omega_m(s) + \tilde{T}_l(s)$$

A block diagram can be constructed as shown in Figure 11a. Note the presence of a feedback path to account for the effect of the motor's velocity on armature current. By moving the load torque summing junction to the left- hand side as shown in Figure 11b, we can see that the motor is in actual fact subject to two inputs: $V_i(s)$ and $\tilde{T}_l(s)$.

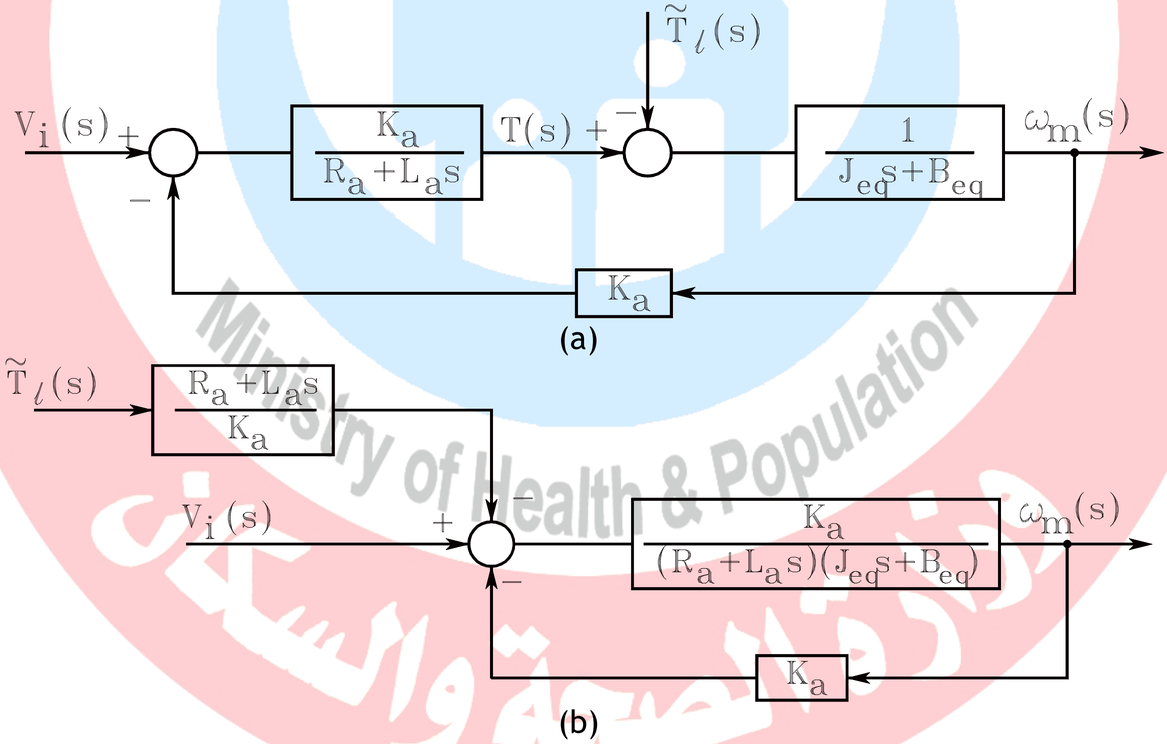


Figure 11. Block Diagram of an Armature-Controlled DC Motor.

Block Diagram of DC Generator

It is a simple matter to demonstrate that the block diagram of Figure 12 represents the dc generator.

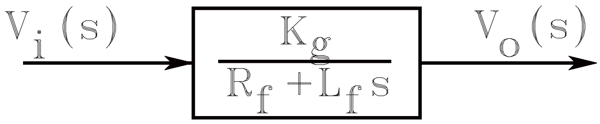


Figure 12. Block Diagram of DC Generator.

Block Diagram of Amplidyne

Figure 13 shows the block diagram representation of the amplidyne discussed earlier.

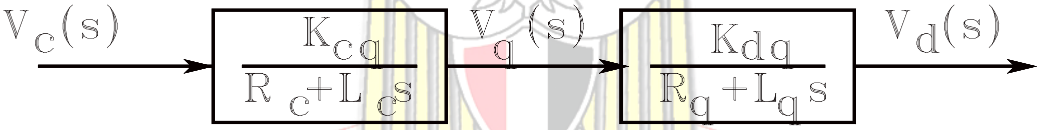


Figure 13. Block diagram of the amplidyne.

Transient and Steady-State Response Analyses

Various methods are available for the analysis of the system performance. In practice, the input signal to a control system is not known ahead of time. In analyzing and designing control systems, we must have a basis of comparison of the performance of various control systems. This basis may be set up by specifying particular test input signals, and by comparing the responses of carious systems to these input signals. Many design criteria are based on the response to such signals or on the response of systems to changes in initial conditions (without any test signals).

The use of test signals can be justified because of a correlation existing between the response characteristics of a system to a typical test input signal and the capability of a system to cope with actual input signals.

Typical Test Signals

The commonly used test input signals are those of step functions, acceleration functions, sinusoidal functions, and the like. With these test signals, mathematical and experimental analyses of control systems can be carried out easily since the signals are very simple functions of time. Which of these typical input signals to use for analyzing system characteristics? The answer is determined by the form of the input that the system will be subjected to most frequently under normal operation. If the inputs to a control system are gradually changing functions of time, then a ramp function of time may be a good test signal. Similarly, if a system is subjected to sudden disturbances, a step function of time may be a good test signal.

For a system subjected to shock inputs, an impulse function may be best. Once a control system is designed on the basis of test signals, the performance of the system in response to actual inputs is generally satisfactory. The use of such test signals enables one to compare the performance of all systems on the same basis.

Transient Response and Steady-State Response

The time response of a control system consists of two parts: the transient response and the steady state response. By transient response, we mean that which goes from the initial state to the final state. By steady-state response, we mean the manner in which the system output behaves as *t* approaches infinity. Thus, the system response can be written as

$$c(t) = c_{tr}(t) + c_{ss}(t)$$

where, $c_{tr}(t)$ is the transient response, and $c_{ss}(t)$ is the steady-state response.

Absolute Stability, Relative Stability, and Steady-State Error

The most important characteristic of the dynamic behavior of a control system is absolute stability, i.e. whether the system is stable or unstable. A control system is stable if, in the absence of any disturbance or input, the output stays in the same state. A linear control system is stable if the output eventually comes back to its equilibrium state when the system is subjected to an initial condition. A linear control system is critically stable if it oscillations of the output continue forever. It is unstable if the output divers without bound from its equilibrium state when the system is subjected to an initial condition. Important system behavior includes relative stability and steady-state error.

Since a physical control system involves energy storage, the output cannot follow the input immediately but exhibits a transient response before a steady-state can be reached. The transient response often exhibits damped oscillations before reaching steady state. If the output of a system at steady state does not exactly agree with the input, the system is said to have steady-state error. This error is indicative of the accuracy of the system. In analyzing a control system, we must examine transient-response behavior and steady-state behavior.

First-Order Systems

Consider the block diagram of a first-order system shown in Figure 13a. Physically this block diagram represents an RC circuit, or a thermal system. A simplified block-diagram is shown in Figure 13b. The input-output relationship is given by:

$$\frac{C(s)}{R(s)} = \frac{1}{Ts + 1}$$

We shall analyze the system responses to such inputs as the unit step, unit ramp, and unit impulse functions. The initial conditions are assumed to be zero. The following analysis is correct, whatever the physical system is, as long as the transfer function is the same.

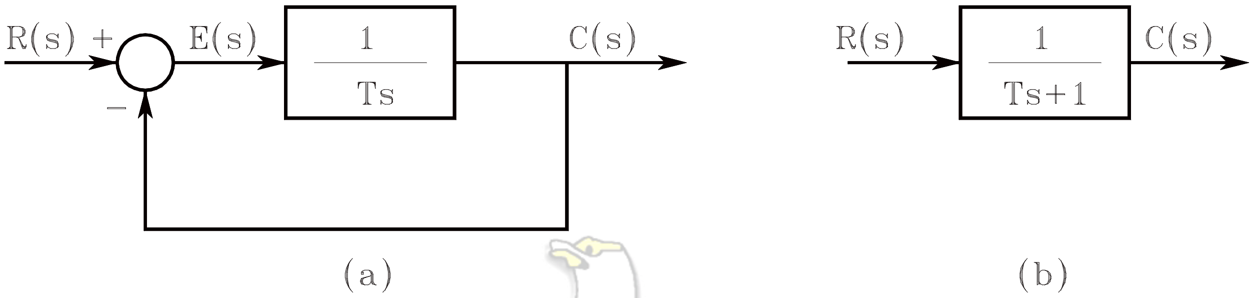


Figure 13. Block diagram of a first-order system.

➢ *Unit-Step Response of First-Order Systems*

Since the Laplace transform of the unit-step function is *1/s*, substituting *R(s) = 1/s* into the above equation, we obtain

$$C(s) = \frac{1}{Ts+1}\frac{1}{s}$$

Expanding *C(s)* into partial fractions gives:

$$C(s) = \frac{1}{s} - \frac{T}{Ts+1} = \frac{1}{s} - \frac{1}{s+(\frac{1}{T})}$$

Taking the inverse Laplace transform of the above equation, we obtain:

$$c(t) = 1 - e^{-t/T}, \qquad for\ t \geq 0$$

The last equation states that initially the output *c(t)* is zero and finally it becomes unity. One important characteristic of such an exponential response curve *c(t)* is that at *t = T* the value of *c(t)* is *0.632*, or the response has reached *63.2%* of its total change. This may be easily seen by substituting *t =T* in *c(t)*.

$$c(T) = 1 - e^{1} = 0.632$$

It is to be noted that the smaller the time constant *T*, the faster the system response. Another important characteristic of the exponential response curve is that the slope of the tangent line at *t = 0* is *1/T*, since

$$\frac{dc}{dt}|_{t=0} = \frac{1}{T}e^{-t/T}|_{t=0} = \frac{1}{T}$$

The output would reach the final value at *t=T* if it maintained its initial speed of response. From the preceding exponential equation we see that the slope of the response curve c(t) decreases monotonically from *1/T* at *t = 0* to zero at *t = ∞*. The exponential response curve is shown in Figure 14.

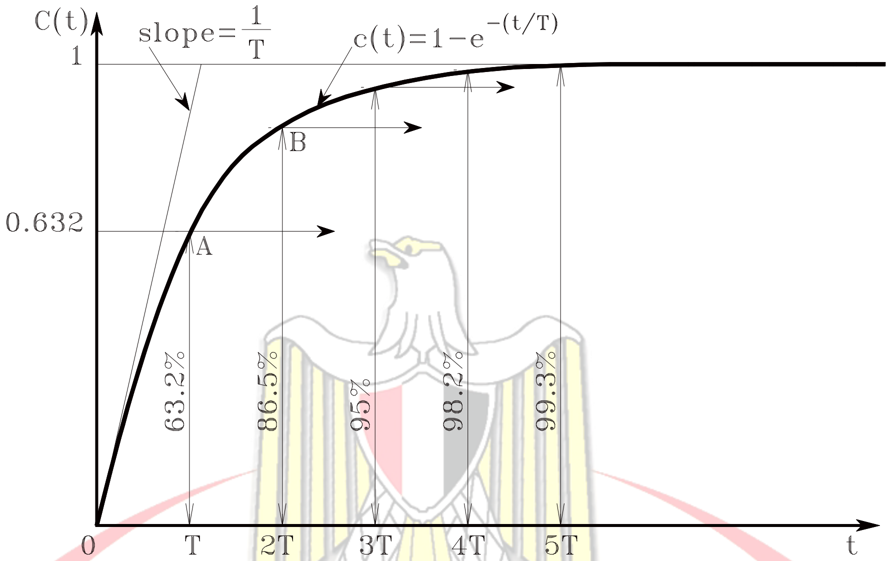


Figure 14. Exponential response curve,

In one time constant, the exponential response curve has gone from *0* to *63.2%* of the final value. In two time constants, the response reaches *86.5%* of the final value. At t = *3T, 4T, 5T*, the response reaches 95%, 98.2% and 99.3%, respectively, of the final value. Thus, for t ≥ 4T, the response remains within 2% of the final value. The exponential equation states that the steady-state is reached only at t = ∞.

In practice, a reasonable estimate of the response time is the length of time the response curve needs to reach and stay within 2% less than the final value, or four time constants.

➢ *Unit-Ramp Response of First-Order Systems.*

Since the Laplace transform of the unit-ramp function is $1/s^2$, we obtain the output of the system of Figure 13 as:

$$C(s) = \frac{1}{Ts+1}\frac{1}{s^2}$$

Expanding *C(s)* into partial fractions gives:

$$C(s) = \frac{1}{s^2} - \frac{T}{s} + \frac{T^2}{Ts+1}$$

Taking the inverse Laplace transform of the last equation, we obtain:

$$c(t) = t - T + Te^{-t/T}, \qquad \text{for } t \geq 0.$$

The error signal *e(t)* is then:

$$e(t) = r(t) - c(t)$$

$$e(t) = T\left(1 - e^{-\frac{t}{T}}\right)$$

As t approaches infinity, $e^{-t/T}$ approaches zero, and thus the error signal $e(t)$ approaches T, or

$$e(\infty) = T$$

The unit-ramp input and the unit output are shown in Figure 15. The error in following the unit-ramp input is equal to T for sufficiently large t. The smaller the time constant T, the smaller the steady-state error in following the ramp input.

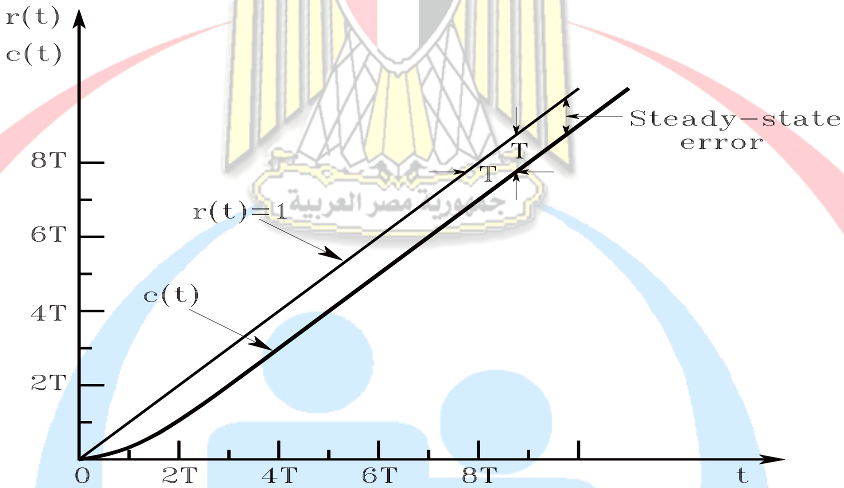


Figure 15. Unit-ramp response of the first order system.

➢ *Unit-Impulse Response of First-Order Systems.*

For the unit-impulse input, $R(s) = 1$ and the output of the system of Figure 13 can be obtained as:

$$C(s) = \frac{1}{Ts + 1}$$

The inverse Laplace transform of the foregoing equation gives:

$$c(t) = \frac{1}{T}e^{-t/T} \qquad \text{for } t \geq 0$$

The response is given by the following equation and is shown in Figure 16.

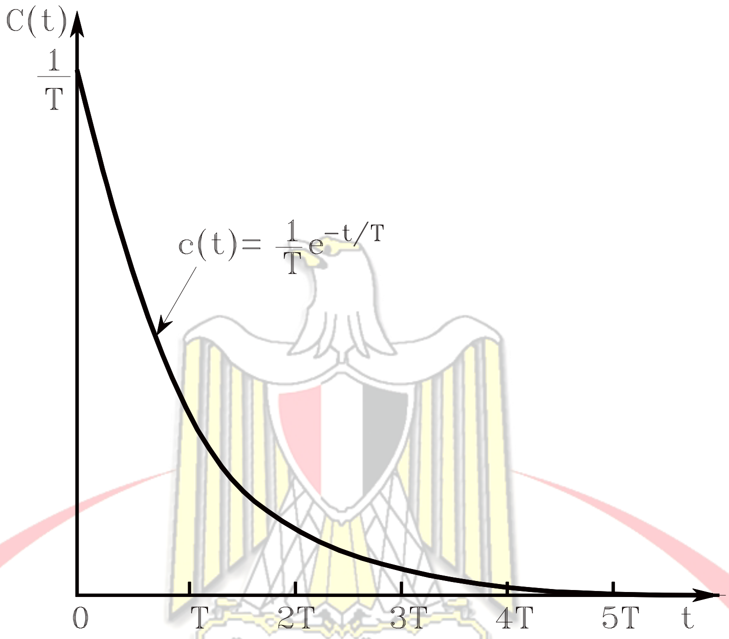


Figure 16. Unit impulse response of the first-order system.

An Important Property of Linear Time-Invariant Systems.

In the above analysis, it has been shown that for the unit-ramp input the output *c(t)* is

$$c(t) = t - T + Te^{-t/T}, \qquad \text{for } t \geq 0.$$

For the unit-step input, which is the derivative of unit-ramp input, the output *c(t)* is

$$c(t) = 1 - e^{-t/T}, \qquad \text{for } t \geq 0$$

Finally, for the unit-impulse input, which is the derivative of unit step input, the output *c(t)* is

$$c(t) = \frac{1}{T}e^{-t/T} \qquad \text{for } t \geq 0$$

Comparing the system response to these three inputs clearly indicates that the response to the derivative of an input signal can be obtained by differentiating the response of the system to the original signal. It can be also seen that the response to the integral of the original signal can be obtained by integrating the response of the system to the original and by determining the integration constant from the zero output initial condition.

References and Recommended Readings

Main References

[1] Ogata K, Yang Y. Modern control engineering. India: Prentice hall; 2002.
[2] Luyben WL. Process modeling, simulation and control for chemical engineers. McGraw-Hill Higher Education; 1989 Aug 1.
[3] Floyd TL, Buchla DM. Electric circuits fundamentals. Pearson/Prentice Hall; 2004.
[4] Charles K. Alexander & Matthew N. O. Sadiku. Fundamentals of Electric Circuits, 4th edition, McGraw-Hill, 2009.
[5] Muhammad H. Rashid. POWER ELECTRONICS HANDBOOK DEVICES, CIRCUITS, AND APPLICATIONS, Third Edition, Elsevier Inc., 2011.

Recommended Readings

[1] Golnaraghi F, Kuo BC. Automatic control systems. Complex Variables. 2010.
[2] Lewis PH, Yang C. Basic control systems engineering. 1997
[3] Nilsson JW. Electric circuits. Pearson Education India; 2008.
[6] Nilsson JW, Riedel S. Electric circuits. Prentice Hall Press; 2010 Jan 13.
[4] Jackson HW, Temple D, Kelly B. Introduction to electric circuits. Englewood Cliffs: Prentice-Hall; 1986.
[5] Ned Mohan. Power electronics : a first course, John Wiley & Sons, Inc., 2012.
[6] Richard C. Dorf, James A. Svoboda . Introduction to Electric Circuits, EIGHTH EDITION, John Wiley & Sons, Inc., 2010.

- **Book Coordinator** **; Mostafa Fathallah**

- **General Directorate of Technical Education for Health**